Melampus

What second-order concepts tell us about the continuum

Abstract

1. The classical concept of the arithmetical continuum gives rise to the second-order Axiom of Completeness. Since this axiom is a second-order principle an excursus on second-order logic investigates the difference between proof in first-order logic and demonstration by means of second-order concepts. Not all methods of inference cohere with an axiomatic system and the nature of the proof that may be expected is an exercise in philosophical mathematics.
2. A phenomenological investigation into the origin of our concept of the continuum shows that the arithmetical continuum is not a priori, that it is the subject of an empirical enquiry and that the standard model of the continuum is an idealised empirical object. The Zeno paradoxes lead to the notion of the continuum as a complete aggregate of limits.
3. The Axiom of Completeness embraces a distinction between the potential infinite and the actual infinite since it adds all real numbers as the limits of potentially infinite sequences.
4. Ordinals are data types and the Cantor set is merely a representing set of the reals in set-theory that cannot be identified with the arithmetical continuum.
5. The view that set theory is the foundation of mathematics is false. The identification of the actually infinite collection of all ordinals with the potentially infinite aggregate of all natural numbers is a paralogism. Set theory without the Axiom of Completeness has no existential statement to call into being real numbers. Set theory is not in fact founded upon the single primitive of set membership. The need to systematically distinguish potentially infinite aggregates from their actually infinite counterparts leads to the introduction of an operation of completeness as the operation of taking a potentially infinite collection and presuming it to form a completed totality. It is also demonstrated that infinite aggregates may have simultaneously different descriptions with different properties. The set of all rational numbers is simultaneously a potentially infinite aggregate subject to the natural order whose order-type upon completion is η and a potentially infinite aggregate subject to the process of generation, whose order-type is ω. The properties of the actually infinite Cantor set and its potentially infinite subset, Fin, which is the set of all finite subsets of ω, lead to a model of the arithmetical continuum as a double tree structure. An axiom of indestructibility of extension that every proper part of an extended portion of space is extended is required. The lattice structure of the continuum and the relation of exponentiation and logarithm upon the lattice reveal the centrality of the skeleton of the continuum to its structure. A model of the continuum which identifies it as the Derived Set by exponentiation of the one-point compactification of the skeleton of the continuum is advanced together with its complete Boolean algebra representation by the Cantor set. The importance of the inexhaustible boundary of the Cantor set in this model is demonstrated.

6. The model of the Derived Set is proven from the Axiom of Completeness in the form of the Heine-Borel Theorem.
7. The null-meagre decomposition is consistent with the model and shows that there are both points of zero measure and infinitesimals in the continuum.
8. The relationship between real numbers in second-order theory and real number generators called ultrafilters in first order set theory is investigated leading to an excursus on ultrafilters, generic sets and forcing arguments. Cohen forcing is described. Concerning the Mahler classification of transcendental numbers theorems are advanced that every Liouville number is a point of zero measure and the transcendental S numbers of the Mahler classification are infinitesimals.
9. A forcing theorem shows that subject to the Axiom of Completeness there is only Cohen forcing in the model of the continuum and that all transcendental numbers of the Mahler classification correspond to ultrafilters produced by Cohen forcing. A theorem that the Completeness Axiom entails the Continuum Hypothesis follows from this, the limitation on the number of data-types and the relationship between the skeleton and the lattice derived by exponentiation from it.
10. There is an excursus on the differences between C19th and C20th mathematics.

Preliminary remarks on the nature of the proof to be expected

There is no purely mathematical resolution of the problem of the continuum

Mathematics is an intensely philosophical activity. This is a truth hidden precisely where it ought to be most clear, and no more so than in the question of what the continuum. There is no purely mathematical resolution of the problem of the continuum.

The arithmetical continuum

In the words of G.H. Hardy: "The aggregate of all real numbers, rational and irrational, is called the *arithmetical continuum*. It is convenient to suppose that the straight line … is composed of points corresponding to all the numbers of the arithmetical continuum, and of no others."[1]

1. The arithmetical continuum is an ideal (one not met with in experience) structure called a straight line.
2. It is also an aggregate of points.
3. The collection of all these points is in one-one correspondence with the aggregate of all real numbers.
4. Real numbers may be partitioned into numbers of two kinds: rational and irrational.

We will call this the "classical conception of the arithmetical continuum".

First and second-order concepts

A formula is first-order if it quantifies over at most individuals. Thus, "All men are mortal" is first-order, because the scope of "all" is "men" and these are individuals. A formula is second-order if it

[1] Hardy, G.H. [1908 / 1967] Chapter 1. §15.

quantifies over properties in addition to individuals. For example, the identity of indiscernibles that states the no two distinct objects can have all their properties in common, is second-order.

Analysis is founded upon the second-order Axiom of Completeness

Mathematical analysis is that branch of mathematics that deals with continuity, motion and rates of change. Analysis is founded upon a second-order principle called the (Dedekind) Completeness Axiom, whose function is to construct real numbers from rational numbers. One approach to the problem of the continuum is to explore the consequences of the Completeness Axiom. *The author is not aware of any attempt to deploy such a strategy since the time of Cantor.*

Analysis is second-order and yet we tell ourselves that all mathematics is first-order

There is a principle at work that the only viable logic is first-order logic. This principle that all logic is first-order entails double-think, for *it is impossible to proceed in analysis upon such a foundation.* Analysis is founded upon second-order concepts and we still have our articles upon measure theory. Mathematicians have not in fact reconstructed analysis upon "constructivist" first order foundations.

To what extent is there a second-order logic?

In contrast to what is known about first-order logic, what is known about second-order logic is so scant that it would be almost correct to say that *there is to all intents and purposes no known second-order logic.*[2]

(1) We know that in contrast to first-order logic, second-order logic, assuming it is consistent, it is not axiomatic and not complete.[3] Additionally, while a first-order logic is incapable of internally distinguishing the (infinite) cardinality of the domain of discourse, a second-order logic can.[4]

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[2] Shapiro [2000] argues in defence of second-order logic, and introduces a deductive rule for it, a rule of substitution. This system is insufficient to perform any useful deduction in analysis. Certainly, the system and the text as a whole has the advantage of demonstrating that the proof of the Schröder-Bernstein theorem, $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$, is second-order. He also states that Cantor's theorem, that the power set of any set is larger than the set itself, is a third-order statement that can be proven in his second-order system. He points out that the well-ordering principle is third-order. These results support the principles advanced in this paper – namely, that first-order logic is not a sufficient language for mathematics and that the claim, so frequently advanced, that all mathematics is first-order set-theory is either plain false, or an attempt at legislation. However, not only is it theoretically impossible to produce a complete second-order system of deduction, but Shapiro's system is so restricted that the claim made here *that there is not all intents and purposes no known second-order logic* is not refuted. See also footnote 5.

[3] **Consistency and completeness:** In any logic it is usual to distinguish deduction from consequence. Suppose A is said to follow from B. Then this means that *if A is true, then B could not possibly be false.* This is a notion of entailment that relies upon the semantical notion of *truth*. We denote this relation by $A \vDash B$. In addition, we have a set of axioms and rules, and by means of these derive in a series of steps *B* from *A*. This is a syntactical notion of entailment and is denoted $A \vdash B$. There are two questions that concern us: (1) is every syntactic deduction also a semantic consequence? That is, does $A \vdash B$ entail $A \vDash B$? In this case the logic is said to be *consistent*. (2) For every valid semantic consequence is there a syntactic proof? That is, does $A \vDash B$ entail $A \vdash B$? In this case the logic is said to be *complete*. Additionally, a logic is said to be *axiomatic* if there is a finite set of axioms for it that render it complete. A logic is also said to be *sufficiently strong*, if it is capable of expressing the operations of addition and multiplication.

[4] **This is the content of the Lowerheim-Skolem theorem** that is itself a consequence of the Completeness theorem for first-order logic; for suppose one writes in first-order logic a theory of real numbers, then that theory also has a denumerable domain. Thus we say that first-order logic is *not categorical* for the set of all real numbers – that is to say, in first-order theory it is not possible to construct a categorical theory of the continuum. Second-order conceptions can make the distinction between infinite domains, so second-order logic may be categorical for the continuum.

(2) As a deductive system, second-order logic is virtually non-existent. For first-order logic we have precise instructions on how to introduce and eliminate the logical operators and quantifiers. For second-order logic, no such deductive system has been constructed. Second-order logic, if that is what we should call it, is a language for the formulation of second-order statements, but not a worked out system of inference. This is something of a paradox. For example, it has been proven that the Dedekind completeness axiom is logically equivalent to (1) the Bolzano-Weierstrass theorem, (2) Cantor's nested interval principle, (3) Cauchy's convergence criterion, and (4) the Heine-Borel theorem. But *in what formal system are these equivalences constructed?* Since every one of these statements is itself second-order, such a deductive formal system *could only be a second-order logic*.[5] In practice, these deductions are made in an unanalysed meta-language that uses a mixture of second-order formulations, natural language reasoning, geometric intuition and algebra.

Not all methods of inference cohere with an axiomatic system

There is a strong preference in mathematical discourse for the axiomatic method – it is the very paradigm of precise, clear and rigorous thinking. However, not all methods of inference cohere within an axiomatic system. There are methods of inference that cannot be formulated in such a way that they can be applied by formal methods within an axiomatic system and the axiomatic method cannot be universal. These methods include:

(1) The Dirichlet Pigeon Hole principle that you cannot fit n items into $n-1$ boxes without putting 2 items into at least one of the boxes. This principle is involved in fixed point theorems, and specifically is employed in the proof of the Mahler classification of transcendental numbers.

(2) Many theorems involve demonstrating the identity of different ways of viewing the same mathematical structure – that is the identity of different descriptions of the same object.[6] Denoting a different description of an underlying mathematical object by the term "intension", then an equivalence of expressions is a construction in the "logic of intentions" – a demonstration that "this is what such and such a description *means* in this other language." While mathematics might be construed as one vast language, it has many sub-languages and equivalences between formulations must be established.

(3) In first-order logic the logical symbols, $\wedge$, $\vee$, $\neg$, $\supset$, ... correspond to conjunctions of natural language: "and", "or", "not" and "if… then" respectively. There is no a priori reason to believe that every inference whatsoever that involves conjunctions can be expressed by first-order logical symbols.

(4) In first-order logic a rule of non-contradiction may be established. However, when proof by contradiction is used in a given mathematical demonstration, there is no a priori reason to suppose that its application is an instance of the first-order rule. For example, a contradiction argument by infinite regress is not a priori a first-order rule.

[5] I shall say that a criterion for an "almost adequate" second-order deductive system is a system in which it a formal proof of the equivalence of the various statements of the Completeness Axiom can be given. It is surprising that the non-existence of such a system has hitherto, to my knowledge, never been noted, while at the same time the equivalence of the five principle statements of the Completeness Axioms is taught in every undergraduate course of mathematics, and has never been in dispute.

[6] An example is Lagrange's Theorem in group theory that the order of a subgroup divides the order of the group: that is, if $H \le G$ then $|H|$ divides $|G|$. This theorem has so far resisted formalisation. Lagrange's theorem is an inference whose purpose is to establish the equivalence of different conceptual ways of looking at an object's (here the group G's) internal structure, where $H \le G$ *already embodies the fact that is said to be its consequence* – namely $|H|$ divides $|G|$.

No axiomatic method to determine the axioms

There is no axiomatic method to determine whether a given set of axioms is the "right" set of axioms. A set of axioms gives the rules governing the manipulation of a primitive concept, but the questions:

1. Is this the right primitive?
2. How many primitives do we need?
3. Is this axiom the right one for this primitive?

are real and meaningful, and transcend the axiomatic method.[7]

A phenomenological enquiry into our primitive experience of the continuum

Noumena and phenomena, or the distinction between appearance and reality

We will restore the language and conceptualisation of Kant, at least for heuristic purposes. What I see, touch, hear, smell and taste with my five empirical senses are deemed phenomena. But such objects only "exist" so long as I see, touch, hear, smell and taste them. I look at the blue wall (for my walls are painted in that colour), but then turn my back upon that wall to greet a guest who is coming into the room by the double doors. Once I have turned my back upon the blue wall, does the sensation of the wall continue to exist in my consciousness? I answer, "no", for the blueness of the blue wall exists only so long as I look at it, as I sense it, as I am aware of it. And yet, does the wall continue to exist? It is perfectly sensible to suppose that it exists. Is the wall that exists when I am not looking at it still blue? No, I think not. I think that the blueness of the blue wall exists only so long as I look at it, but the wall itself might exist even when I don't look at it. It might also exist were I completely annihilated. The same wall could be seen by the people who live in the generations after my death. The blue wall, when I am looking at it, is termed a "phenomenon", and the real wall, the wall that exists whether I exist or no, is termed a "noumenon". This in the language of Kant. Noumena are also called objects-in-themselves, and phenomena are also called empirical objects. The aggregate of all phenomenological objects is called "empirical reality"; the aggregate of all noumena is called "transcendental reality". All men differentiate between appearance and reality, and that is all we one is really saying. The terminology of Kant makes the distinction between appearance and reality clear; but that there is a distinction is all that is required here.

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[7] The language of set theory: this language in which the continuum hypothesis is constructed is said to rest upon a single primitive notion of set membership, denoted $\in$. Once a primitive is given, and its properties are established with a collection of axioms (that is, a set of axioms), then the axiomatic method applies to the deductive consequences of that set. If such is the case, then a theory is capable of an exposition that "follows the axiomatic method". But how does one establish the meaning of the primitive notion? What system of axioms can tell you that you do understand what the primitive means? A related problem is establishing the correct axioms, granted that the primitive is understood. This is particularly moot in the case of set theory, because the "naive" understanding of set theory was shown early on in its formulation to lead to internal contradiction. This meant that a set of axioms had to constructed, and invariably this process lead to a lot of dispute as to whether the chosen collection was the right collection, and whether it may be said to be founded upon some other primitive notion, such as "the iterative concept of set". In what logic or axiom system do we determine whether a given axiom is the right axiom? The formalist reply is that the primitives have no meaning other than that given in the rules and axioms for using them. In what axiom system do we determine whether this formalism is or is not correct?

The continuum as appearance and reality

I experience objects in motion, or, at least, I judge so. I see an ambulance racing past my door – I presume that the *real ambulance* travels across a *real space*, and would do so were I completely annihilated and utterly incapable of perception. So what is my experience of motion? And what is my conception of the real space and real time in which the real motion take place? This invites in the first instance a *phenomenological enquiry* into the subjective experience of space, time and motion. Subsequently, it invites reflection on what real space, time and motion might be, and whether these can at all be known.

The subjective experience of space, time and motion involves the bare notion of continuity but not of the notion of a point

These are the conclusions I draw from reflection upon my experience of space, time and motion.

(1) **Time**. All perception of time for me is in the present, but the present is not experienced as a point or instant, but rather as extended. It is called "the specious present". Furthermore, "extension" here is a metaphor derived from the experience of extended space, since the experience of the specious present is primitive in consciousness. It is a consciousness of the present moment has "having a duration" but the duration of that duration cannot be measured – to ask it to be measured is nonsense. Time appears to be flowing from past through present into future, but "appears" here is a metaphor – rather the mind judges that time flows. The passage of time appears to be correlated with the passing out of experience (the passing out of being) of phenomena and with the coming into experience (the coming into being) of other phenomena. Because all phenomena are experienced within the specious present neither the past nor the future exist in consciousness, and there is no experience in the phenomena of any enduring substratum to the objects that "flit" in and out of experience. Thus much for time.

(2) **Space**. I experience space as extended in three dimensions, but all extended objects are presented to the subjective point of awareness that is located at the point that is situated between the two eyes or eyebrows in the middle of the forehead. (In the terminology of mathematics, it is a projective space; for the artists, it is a kind of perspective.) I do not experience any discrete point in space, for space consists solely of spaces. Space is not constructed from points of no extension, so my primitive experience of phenomenological space is of a continuous space. Space is made of continua or its sub-spaces.

(3) **Space and time.** From the primitive experience in the specious present I obtain the primitive notion of the *flow of time* and the primitive notion of duration. From the primitive experience of space, I derive the primitive notion of extension. By combining the two primitive notions I arrive at the notion of extended passage of time – that is, time-in-itself is a combination of the flow of time with extension. But this transcends the specious present, is not strictly presented to consciousness, and hence is a noumenon, if it exists at all. I also have a faculty of memory, and by use of this faculty I become aware that the configuration of objects now present before me in this experience of extended objects within this specious present is not the same configuration of objects that I remember as being formally present in a specious present that I now judged to have lapsed.

(4) **Motion.** Strictly speaking, I do not experience any object whatsoever as in motion; but unconsciously I infer that a given object has changed its place and has therefore moved from one place to another over the passage of time.

(5) **There is no boundary to consciousness.** Neither time, space, nor motion have any perceptual beginning or end; I experience no boundary between the experience of space and the non-experience of space.
(6) **Realism or Idealism.** From all this experience of phenomena it is possible to conclude either (REALISM) that there is a real space and time in which real objects are moving, and that these real objects (noumena) are what cause in me my perceptions of them at any given experience of the specious present; but I could conclude (IDEALISM) that all space and time exist only in the specious present; that there are no noumena whatsoever, and that all motion is an illusion.

Empiricism and phenomenology

Hume characterised empiricism as "… all our ideas or more feeble perceptions are copies of our impressions or more lively ones", but this is not really what empiricism means for science. The Stanford dictionary states that "phenomenology is the study of structures of consciousness as experienced from the first-person point of view". In this respect, the empiricism of Hume is distinguished from the phenomenology of, say Husserl, only by the somewhat greater emphasis on subjective consciousness in the latter. However, for science empiricism is not just a theory that knowledge is derived from sensation, but also, through the scientific method, an attempt to build a model of nature or systematic rational construction out of experience, which it does by establishing laws and assemblies of laws as theories. Here the term "nature" is deliberately ambiguous, for there are two ways of reading this activity of science. The first is to see it merely as an attempt to find the underlying law like patterns of experience, so that no metaphysical claim about "reality" and its noumena are made. This is called instrumentalism. (The laws of nature are a systematisation of empirical reality.) The second alternative is to suppose that the laws one discovers are the laws of a reality that exists independently of consciousness and describe what happens to the noumena. (The laws of nature are a description of transcendental reality.) Actually, a mathematician is not really forced to decide between these idealist and realist options; he can even be a naive realist and presume that he is studying real things, for naive realism is consistent with both idealism and realism.

Primitives from the phenomenological enquiry

The phenomenological enquiry has revealed the following notions as primitives[8] of subjective experience without recourse to experiment as such.

> **Extension.** Space is an extended continuum.
>
> **Continuity.** Space is composed of space or continua are composed of continua.
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> **The flow of time** as consciousness of experience passing out of being and coming into being.
>
> **Memory** of that which is not now present.

The arithmetical continuum is not a priori and is the subject of an empirical enquiry

By combination of primitive characteristics, we obtained further concepts

> **Duration.** Time has a duration of past, present and future; it is continuous and extended.
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> **Motion.** Objects in space and time change place and are judged to be at rest or in motion.

[8] The term "primitive" is used in two senses in this essay. Firstly, to denote a primitive notion of subjective consciousness; and secondly as a primitive concept that forms the foundation of a theory. Context determines which notion is being used.

These latter two principles are outside the scope of the direct phenomenological experience, and are tinged with a realist "flavour". In other words, an idealist could deny both and argue that all space, time and motion are illusions merely – that "All is One".[9] The arithmetical continuum is not given in phenomenological experience, where only the bare concept of continuity is met. Hence this concept that we encounter in analysis is and always has been an empirical construction of mathematical science. **The structure of the arithmetical continuum is not given a priori; it is an empirical question.** If science so wishes, it may alter the construction of the arithmetical continuum.

The standard model of the continuum is an idealised empirical object

So we must ask, how is it that science has empirically arrived at the standard model of the arithmetical continuum?

(1) **Measurement.** We take a stick, *walk about*, and *discover* that objects can be measured and that objects that once measured do not change their size.

(2) **Points.** We make marks upon the stick and transform it into a ruler; these marks transform in our minds into points and boundaries, which are "ideal" objects in the sense that *they do not exist in experience*, but are concepts introduced in science to regularise and explain the uniformity of experience.

(3) **Ratio**. The uniformity of measurement enables us to compare the length of one object to another.

(4) **Locus.** We start to construct a geometry of our space, and discover that between two points it is possible to travel by many different routes; we form the notion of a shortest path between two loci. We arrive at circular movement, and subdividing the circle, at a unit of circular measure.

(5) **Geometry.** We envisage ideal geometric objects such as a circle, square, hexagon and so forth.

(7) **Incommensurability.** We discover that certain ratios are incommensurable with each other. We arrive at the concepts of a rational and irrational number.

(8) **Idealisations.** To the primitives of subjective experience, we add primitives of empirical science: point or boundary, congruence of figure, shortest distance or straight line, uniformity (comparison of the same object at different times, or of two objects at the same time reveals that they remain congruent), measure (uniform objects may be compared to a standard ruler), ratio, incommensurable ratio, rational and irrational numbers, velocity (if things are in motion, then it is possible to compare of measure of change of place with change of time).

* These primitives of mathematical science are all *idealisations*: that is to say, no point is ever experienced: no two objects are ever known to be exactly congruent; and no shortest path is ever traversed. M**athematical empirical science has constructed an idealised picture of nature that is never encountered in experience**. Furthermore, it has done so without being explicit as to whether this idealised nature exists in transcendental reality (realism) or is a fiction introduced to systematise the uniformity of experience (instrumentalism/idealism). And yet, we still have not arrived at the classical notion of the arithmetical continuum.

The consistency, connectedness and homogeneity of space

There are other primitives. Experience shows us that space does not behave in odd ways. Space does not "jump about" and "reappear" in different places; it does not behave as it is sometimes made to do in certain science fiction stories. One does not find a larger space within a smaller space or the same

[9] Parmenides and Zeno and the Eleatic School.

space within itself. It is not a simple matter to characterise all the primitive properties of space, or to say which of these belong purely to primitive consciousness, and which are actually empirical concepts learned through some elementary and possibly pre-cognitive experiment.

Zeno paradoxes lead to the notion of the continuum as a complete aggregate of points

> **Conjecture**. ATOMISM: Space and time are composed of extended atoms each of finite measure. Any given continuum comprises a finite number of atoms of space. The atoms of space and time are monads smaller than which it is not possible to divide.[10]

(1) This conjecture cannot be ruled out as incoherent, but does entail a number of conclusions that could impose severe restrictions upon mathematics as a science. It falls foul of the **Zeno paradox of the stadium**. If there are atoms of space and time, then there is an absolute slowest rate of change of place (motion) of one atom of space in one atom of time. However, let two objects *A* and *B* be traveling towards each other and let the relative motion between them be one atom of space per one atom of time. Let there be a third object stationary. Then relative to that third object each of *A* and *B* are travelling at the rate of half an atom per one atom of time. Hence there cannot be an absolute slowest rate of change.

(2) Given two adjacent atoms – what is the manner in which they "touch" each other? Suppose the boundary has no measure, then the boundary is an atom of no measure, and what we thought were atoms are not atoms. Suppose the boundary has a measure, and hence is an extended continuum. Then what is the measure of that continuum? If it is smaller than the measure of the atom, then we did not have atoms. If it is larger than the atom, then the two atoms overlap, and they are not atoms.

Conclusion: Not that the notion of a space as composed of a discrete, finite number of atoms is incoherent, but merely that *it is not the notion enshrined in the classical concept of the arithmetical continuum*.[11] What in fact the standard model has done is incorporate yet another notion into the concept.

> **Completeness**
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> Every ratio, commensurable or incommensurable, exists. It is possible to go on sub-dividing any continuum *ad infinitum*. There are no atoms of finite commensurable measure of space and time. Space and time as aggregates of continua are continuous.

This is what is meant when Hardy writes, "The aggregate of all real numbers, rational and irrational, is called the *arithmetical continuum*." Without this concept there would be equations that have no solution, for unless $\sqrt{2}$ exists then the equation $x^2 = 2$ has no solution.[12]

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[10] This proposition has been advanced by Hume in *The Treatise*, and is currently in vogue among scientists and philosophers.

[11] I have already observed that the structure of the continuum is an empirical problem, so if as a matter of fact science decides to replace the standard model by another one, then so be it. However, the onus shall be on the proponents of such a view. For example, atomists shall be required to demonstrate how they overcome the two paradoxes above and the practical benefits of the substitution.

[12] **A pragmatic approach to these questions may be best.** Is it possible that space should be composed of atoms, and time only a specious present? This is consistent with the phenomenological concepts of space and time given above, and avoids the Stadium paradox. However, it is a thorough going idealism of the Eleatic variety, and as such does not serve as a foundation for mathematical science, for under it, all motion is an illusion. One conclusion that I personally draw is as follows: we never really know what transcendental reality is "like". From a pragmatic point of view, we wish to

The Axiom of Completeness

The potential infinite

Completeness is not a primitive notion, for it rests upon at least one other concept that is more primitive – that of the potential infinite.

> **The potential infinite:** no matter how large a number we have reached it is always possible to count to a higher one by adding one more. Counting is inexhaustible.

As Aristotle demonstrated, the potential infinite is a sufficient concept to solve the Zeno paradoxes of the Dichotomy and Achilles and the Tortoise. We take the Dichotomy as an example:

> **The Dichotomy of Zeno**
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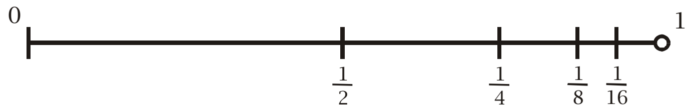
> We suppose that Achilles is running from 0 to 1 across the unit interval.



> Firstly, he gets half-way, and then three-quarters of the way, and so on. Since he cannot pass through an infinite number of points, he can never reach 1.

In running for 0 to 1 there is no actual division of the line; the points of division are idealised points only and exist merely in conception, not in real space. The line is divided in thought only, and it is always possible to divide the line one more time, even though actually, and in practice, one only divides it a finite number of times. The Dichotomy really poses no greater problem than is already posed by motion itself. That motion occurs and that objects change their place is a judgement of the mind based on its primitive phenomenological experience. No one really knows whether in reality (that is transcendental reality, independent of subjective consciousness) any objects move, or if they do so, how they move, and assuming that they do move, whether they traverse an aggregate of points. The arithmetical continuum is strictly an empirical construct. It remains to determine whether it is consistent, but the Dichotomy does not show its inconsistency. The potential infinite does legislate against the atomic theory of space and time, since according to that theory the line cannot be divided *ad infinitum*. So the potential infinite does say that the line is continuous; however, it is not enough to define its completeness. For that, we need an additional concept.

The arithmetical continuum requires also the notion of an actual infinity

> **The actual infinite:** the entire process of counting forms a completed totality. A completed collection of infinite objects is given actually in its entirety.

The potential infinite is not sufficient to establish the completeness notion – *that every ratio, commensurable or incommensurable exists in the continuum.* Subdivision of the line by commensurate measure is insufficient to generate a single irrational number. This is the real problem of the Dichotomy, for no matter how I divide that line I cannot construct an irrational measure. There is, however, one possibility that needs to be considered.

> **Conjecture**

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explain our empirical reality, discover its laws and formulate them in a complete mathematical language. Hence, we live with and use the concept of the arithmetical continuum as a practical tool, while remaining silent as to whether real space, independent of human conscious, is actually composed of continua and whether points of zero measure really exist in that transcendental space.

Every (incommensurable) real number may be identified with the potentially infinite sequence that generates it. For example, by bisection: $\sqrt{2} = \langle (1,2),\ (1,1.5),\ (1.25,1.5),\ (1.375,1.5), \ldots \rangle$

This demonstrates $\sqrt{2}$ as a convergent sequence of nested intervals.[13]

This conjecture is in vogue in contemporary philosophy especially in the constructivist and intuitionist schools, but it poses problems.

1. It is not possible to say whether two distinct convergent sequences converge upon the same real number. The notion of limit is excluded from this conception. Hence, there are as many $\sqrt{2}$ as there are convergent sequences that approximate it.
2. As the above illustrates, $\sqrt{2} = \langle (1,2),\ (1,1.5),\ (1.25,1.5),\ (1.375,1.5), \ldots \rangle$ is an set of ordered sequences of nested intervals. Hence, it is not a point nor can be identified with a point. It is not an element of the arithmetical continuum.

Under this conjecture the symbol $\sqrt{2}$ is without justification; the conjecture makes no sense of the identity of convergence of sequences. It is not a way of constructing an alternative to real numbers; real number generators cannot be substituted for their limits.[14]

The completeness axiom adds all real numbers as the limits of potentially infinite sequences

That both the potential and actual infinite are involved in the classical theory of the continuum is self-evident. Consider the expression:

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

Here on the left hand side we see a convergent sequence of rational numbers that is a potential infinity, indicated by the notation $n \to \infty$. The symbol ∞ denotes the potential infinite. The fact that the entire sequence is taken as a totality is denoted by the expression lim. On the right hand side the limit has been taken and a symbol has been introduced to denote the unique real number to which that sequence converges. The structure of the arithmetic continuum must accommodate both sides of such an expression.

1. There must be a potentially infinite part to the continuum wherein sequences converge *ad infinitum*.

[13] This conjecture has immediately introduced **the language of set theory**. This is historically how set theory was developed. Firstly, it was a tool for expressing the notions of sequence and convergence. The expression above uses the notion of an ordered set, and the notion of an open set expressing an interval. So we have immediately experienced a "jump" in the heuristic, for the moment we talk of generating real numbers we have the language of set theory. Since set theory requires a new primitive, we introduce that now. **Set membership**: the relation of an element to any aggregate to which it belongs is given. The element is said to be a member of the set to which it belongs.

[14] **The calculus requires the actual infinite**: The problem of convergent sequences arose historically after or at least in the context of the invention of the calculus. It is standard analysis that the construction of the real number, that is, the notion of limit, precedes definition of rate of change, or the derivative. So we do not require a separate argument from the calculus to the arithmetical continuum, for they are one and the same. However, historically, Newton effectively ducked the problem of the completed infinity with his concept of a fluxion; Leibniz was explicit: the calculus requires $\dfrac{dy}{dx}$ and dy and dx are "infinitesimals", the meaning of which shall be considered below, but certainly subsumes the notion of an actual or completed infinity. Thus, no actual infinity = no calculus = no analysis. Throw out the real number, and you throw out analysis – that is, you chuck out the bath with the bath water.

2. The continuum must also contain all limit points of all convergent sequences.

Firstly, we need an axiom, or primitive, that will construct the real numbers as required. There are at least five separate equivalent forms of this axiom. This is the completeness axiom in the form provided by Cantor:

The completeness axiom: Cantor's nested interval principle

Given any nested sequence of closed intervals in $\mathbb{R}$, $[a_1,b_1]\supseteq[a_2,b_2]\supseteq\ldots\supseteq[a_n,b_n]\supseteq\ldots$, there is at least one real number contained in all these intervals: $\bigcap_{n=1}^{\infty}[a_n,b_n]\neq\varnothing$.

About this axiom: (1) It is second order: "For all nested sequences…"; (2) It makes an existential statement: "There exists at least one real number…" hence constructs a real number, calling it into existence, and thereby constructs the continuum.[15]

Definition, separable

A separable space has a countable dense subset.

Separability expresses the manner in which real numbers are constructed out of the sequences of rational numbers that approximate them. Real numbers do not appear "out of nowhere", but emerge on closure of a dense subset, where closure means the addition of all limit points. Separability may be shown to follow from the Axiom of Completeness. However, referring to the distinction between the potential and actual infinite:

1. The potentially infinite part of the continuum corresponds to a countable dense potentially infinite subset of the continuum that is isomorphic to the potentially infinite set of all rational numbers, $\mathbb{Q}$.
2. To take the closure of this dense subset means to add all its limit points. This closure is equivalent to forming the complete, actual infinite sequence of any potentially infinite sequence of rational numbers. This adds all real numbers.

A point has no measureable size

Examining again the words of G.H. Hardy:

> It is convenient to suppose that the straight line … is composed of points corresponding to all the numbers of the arithmetical continuum, and of no others.[16]

What is the size of a point? What is its extension, magnitude or measure?[17]

Paradox of large (Zeno): Suppose that a point has a measureable magnitude. Then, since a closed interval of finite magnitude is an aggregate of an actually infinite collection of such points, that closed interval must have infinite size, which is a contradiction.

---

[15] This axiom is apparently circular in that the set of reals, $\mathbb{R}$, appears to be presupposed. In fact, as is generally known, the reals are constructed from the rational numbers, and the circularity is apparent only.

[16] It is a synthetic connection to identify real numbers with points; the bare conception of a real number does not contain within it any notion that it could stand for or label a point. It may be customary to define the arithmetical continuum as "the aggregate of all real numbers", but Hardy at least sees the need to explicitly connect this aggregate with "points corresponding to all the numbers of the arithmetical continuum".

[17] The same question might be asked of a real number: does a real number have a size? The obvious answer to the latter is "no" – for a real number may be used to measure a magnitude, but it is an abstract or ideal object itself that has no size. In fact, this question alone illustrates that the connection between a real number and a point is synthetic, for while it is absurd to ask whether a real number has a magnitude, it is not absurd to ask it of a point.

Paradox of small (Zeno): Suppose that a point has no magnitude. Then since an infinite collection of objects of zero magnitude has zero magnitude no interval could be made up of points.

> **Conjecture: there exist infinitesimals**
>
> An infinitesimal object is an object that has no measureable magnitude, but is not zero in magnitude. It is an object so small that it cannot be distinguished from zero by any available measure from within the theory of numbers given.

The Paradox of large demonstrates that a point must have zero magnitude. The Paradox of small poses a challenge that should be overcome. One possibility is that the rules for multiplication of infinite collections are different from those of finite collections, so that an infinite number of zeros can equal a measure. This is fundamentally the hope enshrined in the Continuum Hypothesis – that the very large collection of points – of cardinality continuum - is so a large infinity that, indeed, continuum many points can fill up a line. Ultimately, this this is the only option that saves the classical conception of the continuum, but we shall see that first-order set-theory is incapable of expressing it. It will be a part of the theory presented here that **there exist infinitesimals**.

* To our classical concept of the arithmetical continuum we have added by way of clarification:

5. A point has no measureable magnitude.
6. The set of rational numbers forms a countable dense subset of the set of all reals.

Ordinals and cardinals

Ordinals

Ordinal numbers are expressed in natural language using the terms, “first”, “second”, “third” and so forth. These contrast with natural counting numbers, “one”, “two”, “three” and so on. Ordinal numbers relate to position within an order, and natural numbers relate to the size of a collection, and they are in the context of set theory called cardinal numbers. The answer to the question, “What was the runner’s position in the race?” is an ordinal number; the answer to the question, “How many people entered the race?” is a cardinal number. In set theory an ordinal is defined to be a transitive set all of whose members are also transitive sets. (A set is transitive if every member of it is a subset.) Finite transitive sets can be arranged in a well-ordered sequence:

$$
\begin{array}{lll}
0 & =\varnothing & \\
1 & =\{\varnothing\} & \\
2=\{0,1\} & =\{\varnothing\}\cup\{\{\varnothing\}\} & =\{\varnothing,\{\varnothing\}\} \\
3=\{0,1,2\} & =\{\varnothing,\{\varnothing\}\}\cup\{\{\varnothing,\{\varnothing\}\}\} & =\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\} \\
4=\{0,1,2,3\} & =\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\}\cup\{\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\}\} & =\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\},\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\}\}
\end{array}
$$

This sequence goes through a potential infinity: $\{0, 1, 2, 3, \ldots, n, \ldots\}$, which is a set of finite successor ordinals. The trick of set theory is to define a limit ordinal to be neither the null set nor a successor ordinal, to denote this ordinal by ω which is also defined to be the order (type) of the set $\{0, 1, 2, 3, \ldots\}$. We have ordinal addition and ordinal exponentiation:

$$\omega+\omega=\omega\cdot 2 \qquad\qquad \omega\cdot\omega=\omega^2$$

It can be shown that the aggregate of all ordinals is well-ordered, that is, may be placed in a single list of successive ordinals:

$$0,1,2,3,\ldots,n,\ldots\omega,\omega+1,\ \omega+2,\ \ldots,\omega^2,\ \omega^2+1,\ \ldots,\omega^3,\ \ldots,\ \omega^\omega,\ \omega^\omega+1,\ \ldots,\ \omega^{\omega^\omega},\omega^{\omega^\omega}+1,\ \ldots$$

This aggregate is obviously not bounded above (it carries on indefinitely), and the entire aggregate cannot be a set (the content of the Burali-Forti paradox) – we say it is a "proper class".

What are ordinals really?

In set theory the notion of successor ordinal may obscure the real nature of what an ordinal is. Thus, it is important to emphasise:

Ordinals are just data types

Ordinals are equivalence classes of data-types.

Some examples:

Ordinal	Data-type
n	A one column or one row matrix, a vector or co-vector.
$1+n$	A singleton set followed by a one row matrix with n entries.
$n+1$	A one row matrix of n values followed by a singleton matrix.
$n+n$	The addition of two column or row matrices each with n entries.
$m\cdot n$	A matrix with m rows and n columns.
$l\cdot m\cdot n$	A three-dimensional data object.
ω	A column or row vector that is imagined to have an actually infinite number of entries that may be put into one-one correspondence with the natural numbers.
$1+\omega$	A singleton set followed by a data object of type ω.
$\omega+1$	A data object of type ω followed by a singleton set.
$\omega\cdot\omega=\omega^2$	A square matrix with actually infinite, ω, rows and columns.

And so on. The rules for addition, multiplication and exponentiation of ordinals then follow from a decision on how to take their ordering and pairing. The pairing is obviously a product of sets, and the rule follows the rule of "lexographic" ordering.

Lexographic ordering

What follows are some examples to illustrate the meaning of "lexographic" ordering. In these examples a similarity relation is denoted by '=' . It is a relationship between data-types that means that two apparently different data-types are instances of one underlying type.

(1) $1+\omega=\omega$

The left-hand side represents a singleton set followed by a matrix with ω entries. Since the second set is infinite, we can just reorganise the whole structure into a matrix with ω entries.

(b) $\omega+1\neq\omega$

In ω there is no largest element but in $\omega+1=\omega\cup\{0\}=\{\omega,0\}$ the largest element is ω. So the two types are genuinely different.

(c) $2\cdot\omega=\omega$

The elements of $2\cdot\omega$ are $(0,0)<(1,0)<(0,1)<(1,1)<(0,2)<\ldots$ which is of the same as a data structure of order type ω.

(d) $\omega\cdot 2\neq\omega$

Elements of $\omega \cdot 2$ are $(0,0)<(0,1)<(0,2)<\ldots<(0,n)<\ldots(1,0)<(1,1)<(1,2)<$.

So there are elements in $\omega \cdot 2$ that are preceded by infinitely many elements and so they are different data-types.

Order-types

There are fundamental order-types. Firstly, all the finite ordinals represent different data structures. Secondly, there is a data-structure that essentially has ω entries. But there are other kinds of data-structure that have so far not been considered:

Set	Order type	Description
$\mathbb{N}$	ω	The set of natural numbers has a first element, but no last
$\mathbb{Q}$	η	The set of rational numbers is compose only of rational numbers, but has the feature that it has no first or last element. Between any two rational numbers there is another rational number. It is said to be dense. Nonetheless, η is countably infinite.
$\mathbb{R}$	λ	The set of real numbers it the closure of $\mathbb{Q}$. Like $\mathbb{Q}$ it has no first or element, but it is also not countably infinite.

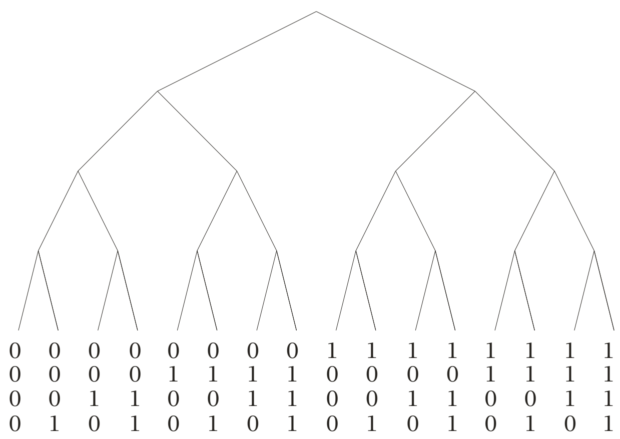
The proper class of all ordinals can be well-ordered:

$$0,1,2,3,\ldots,n,\ldots\omega,\omega+1,\ \omega+2,\ \ldots,\omega^2,\ \omega^2+1,\ \ldots,\omega^3,\ \ldots,\ \omega^\omega,\ \omega^\omega+1,\ \ldots,\ \omega^{\omega^\omega},\omega^{\omega^\omega}+1,\ \ldots$$

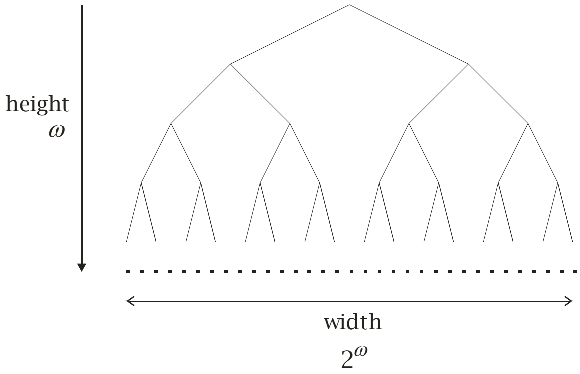
There is the question: where in this list do η and λ appear? The only thing we can say about them is $\omega<\eta<\lambda$.

The representing set of the reals in set-theory

That every real number has an infinite decimal expansion is well-known. The equivalence of the decimal to the binary is also a straightforward matter, and for theoretic reasons, it is simpler to use the binary expansion. Thus, we can generate as a tree, the set of all real numbers.



This tree must continue through actually infinitely many iterations in order to generate the binary expansion of every real number. Each real number is an actually infinite sequence of 0s and 1s of length ω. So each branch of the actually infinite complete tree is of length ω, and the base has 2^ω separate branches. We say that the *height* of the tree is ω and the *width* 2^ω.



In this diagram the dots indicate that the braches are actually infinite in length (height).

The coding

By taking the cross product of the set $\mathbf{2} = \{0,1\}$ with itself, we obtain:

$$\mathbf{2}^2 = \{0,1\}^2 = \{0,1\} \times \{0,1\} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

(Note that the set $\{0,1\}$ is represented by the bold face number **2**.) The ordered pairs are written in column form for convenience, which also displays them as vectors. The structure is a vector space. By iteration of this process any finite number of times, *n*, we obtain a finite tree with 2^n end points corresponding to vectors of size n. These may be equated with a binary code for 2^n finite numbers: For example, given $n = 4$, we may write: $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2} + \frac{1}{4} + \frac{0}{8} + \frac{1}{16} = \frac{13}{16}$. The choice of coding is an interpretation, for we could use: $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{3} + \frac{1}{9} + \frac{0}{27} + \frac{1}{81} = \frac{55}{81}$ and so on. It is a remarkable fact that by taking the complete actually infinite product of $\{0,1\}$ with itself we obtain *all the real numbers* regardless of what coding we actually start with. That is, the set of reals is represented by the set $\mathbf{2}^{\omega}$, which is called the **Cantor set**: $\mathbf{2}^{\omega} = \{0,1\}^{\omega} = \underbrace{\{0,1\} \times \{0,1\} \times \{0,1\} \times \ldots \times \{0,1\}}_{\omega \text{ times}}$. To say that the Cantor set represents the real numbers is simply to say that *there are as many objects in the Cantor set as there are real numbers and the real numbers may be placed in one-one correspondence with them.* It does not say that the Cantor set *is the set of real numbers* – a statement that is palpably false, as will be demonstrated subsequently.

Comparing the sizes of sets – ordinals and cardinals

At a dinner party, the number of knives will equal the number of forks if, and only if, for every knife there is a fork.

> **Hume's principle.** The number of *F's* is equal to the number of *G*'s iff there is a one-one correspondence between the *F*'s and the *G*'s.

We write $X \sim Y$ iff there is a one-one mapping from X onto Y. X and Y are said to be *equipotent* or *numerically equivalent*. Besides the term "equivalent" the terms "equipollent" and "equinumerous" are also used. It is a result that $X \sim Y$ is an equivalence relation, and on that basis establishes a well-ordered sequence of cardinal numbers, representing the size of a set.

Theorem (Cantor)

The set of rational numbers $\mathbb{Q}$ is equipotent to the set of natural numbers $\mathbb{N}$: $\mathbb{Q} \sim \mathbb{N}$.[18]

Definition, diagonalisation

The argument used by Cantor in this theorem is known as *diagonalisation*.

Definition, aleph

$\text{card}\,\mathbb{N} = \aleph_0$. That is, $\aleph_0$ denotes the number of elements in a countably infinite set.

Cantor's "anti-diagonalisation" theorem[19]

The set of real numbers, $\mathbb{R}$, is not equipotent to the set of natural numbers, $\mathbb{N}$.[20]

---

[18] **Proof**. Write the rational numbers as follows

$$\begin{array}{ccccccc}
\frac{1}{1} & \rightarrow & \frac{2}{1} & & \frac{3}{1} & \rightarrow & \frac{4}{1} \quad \cdots \\
 & \swarrow & & \nearrow & & \swarrow & \\
\frac{1}{2} & & \frac{2}{2} & & \frac{3}{2} & & \\
\downarrow & \nearrow & & \swarrow & & & \\
\frac{1}{3} & & \frac{2}{3} & & & & \\
 & \swarrow & & & & & \\
\frac{1}{4} & & & & & & \\
\downarrow & & & & & &
\end{array}$$

By taking the sequence along the zig-zag diagonals as shown every rational number may be placed in correspondence with the set $\mathbb{N}$. This shows $\text{card}\,\mathbb{Q} \le \text{card}\,\mathbb{N}$, but since $\mathbb{N} \subset \mathbb{Q}$ then $\text{card}\,\mathbb{N} \le \text{card}\,\mathbb{Q}$, whence $\text{card}\,\mathbb{Q} = \text{card}\,\mathbb{N}$ and $\mathbb{Q} \sim \mathbb{N}$. The sequence contains infinite repeats of every rational number, for example $\frac{1}{2} = \frac{2}{4}, \frac{3}{3} = \frac{4}{3}$ so in this sense there are more natural numbers than rational numbers. This apparent paradox is resolved by the observation that an infinite set is not altered in size by the removal from it of any collection, be it finite or infinite. Every infinite set is equipotent to a subset of itself.

[19] Sometimes "diagonalisation" is used for this argument. I use "anti-diagonalisation" to distinguish it from the proof of the previous theorem. Anti-diagonalisation shows that sets *cannot* be paired off, whereas diagonalisation shows that they can.

[20] **Proof.** Consider the interval $(0,1) \cong \mathbb{R}$. Suppose there is a bijection from this set to $\mathbb{N}$. Then $(0,1) \cong \mathbb{R}$ is a countable set; so let $l_1,\ l_2,\ \ldots..,\ l_i,\ \ldots..$ be an enumeration of it; that is, a countably infinite list of all elements of $(0,1) \cong \mathbb{R}$ Each element l_i may be expressed as a decimal of the form $l_i = 0.\ a_{i,1}\ a_{i,2} \ldots.\ a_{i,j} \ldots.$ where each $a_{i,j}$ is an integer $a_{i,j} \in \{0,1,2,3,\ldots 9\}$. Construct a number $b \in (0,1)$ such that the decimal expansion of b is

$$b = a_{11} + 1(\text{mod}\,10)\ a_{12} + 1(\text{mod}\,10) \ldots\ldots\ldots a_{ii} + 1(\text{mod}\,10) \qquad (1)$$

$$\begin{array}{cccccccc}
l_1 = & 0. & a_{11} & & a_{12} & \ldots & \ldots & a_{1j} \quad \ldots \\
 & & & \searrow & & & & \\
l_2 = & 0. & a_{21} & & a_{22} & \ldots & \ldots & a_{2j} \quad \ldots \\
 & & & & & \searrow & & \\
 & & \ldots & & \ldots & & \ldots & \ldots \\
 & & & & & & \searrow & \\
l_i = & 0. & a_{i1} & & a_{12} & \ldots & \ldots & a_{ij} \\
 & & & & & & & \quad \searrow
\end{array}$$

Notation

It is usual to denote the cardinality of the real numbers (which is the cardinality of the continuum) by $\mathfrak{c}$.

The anti-diagonalisation argument establishes: $\mathfrak{c} > \aleph_0$. It is usual to assume that $\mathfrak{c}$ is a cardinal, though we cannot actually prove this within first-order set theory. Possibly there are many cardinal numbers, and they cannot be compared, but a further theorem, known as Hartog's theorem establishes that there is a well-ordered succession of cardinal numbers which we write: $\aleph_0, \aleph_1, \aleph_2, \ldots$. Just as we have multiplication and exponentiation for ordinal numbers, so we have multiplication and exponentiation for cardinal numbers. What the above two theorems have established is that for $\aleph_0$:

$$\aleph_0 + \aleph_0 = \aleph_0 \qquad\qquad \aleph_0 \cdot \aleph_0 = \aleph_0$$

These rules generalise to all alephs. So what the alephs do is bunch together the infinite ordinal numbers in to great classes comparable by size. In the succession

$$0,1,2,3,\ldots,n,\ldots\omega,\omega+1,\ \omega+2,\ \ldots,\omega^2,\ \omega^2+1,\ \ldots,\omega^3,\ \ldots,\ \omega^\omega,\ \omega^\omega+1,\ \ldots,\ \omega^{\omega^\omega},\omega^{\omega^\omega}+1,\ \ldots$$

all the ordinals from ω upwards are said to belong to the second number class, and the cardinality of each member is $\aleph_0$. We define ω_1 to be the least uncountable ordinal, so it is the first member of the Third Number Class to be placed after all the ordinals in the succession above.

Ordinal and cardinal exponentiation

Ordinal exponentiation and cardinal exponentiation do not agree:

Ordinal exponentiation	$2^\omega = \underbrace{2\cdot 2\cdot 2\cdot \ldots}_{<\omega\text{ times}}$	exponentiation by potential infinity
Cardinal exponentiation	$2^\omega = \underbrace{2\cdot 2\cdot 2\cdot \ldots}_{\omega\text{ times}}$	exponentiation by actual infinity

By ordinal exponentiation we never leave the second number class and $\mathrm{card}\left(2^\omega\right) = \left|2^\omega\right| = \aleph_0$. By cardinal exponentiation we take the completed, actual infinite power set of 2^ω, and we have $\mathrm{card}\left(\mathfrak{c}\right) = 2^{\aleph_0}$. It is equivalent to taking the power set of the set $\mathbf{2} = \{0,1\}$. We may write $\mathbf{P}(X)$ for the operation of forming the power set of a set *X*. The cardinality of ω_1 is $\mathrm{card}\left(\omega_1\right) = \aleph_1$. Of course, the very problem we are addressing is the question: where on the scale of alephs does $\mathfrak{c} = \mathrm{card}\left(\mathbf{P}(\omega)\right) = 2^{\aleph_0}$ appear, assuming that $\mathfrak{c}$ is a cardinal? The Continuum Hypothesis is $\mathfrak{c} = \mathbf{2}^{\aleph_0} = \aleph_1$.

The number b is constructed by adding 1 (mod 10) to each of the diagonal elements as shown in the above array. By hypothesis, $b = l_k$ for some *k*. That is $b = l_k = a_{k,1}\,a_{k,2}\ldots a_{k,k}\ldots$. But by (1): $b = a_{k,1}+1\left(\mathrm{mod}\,10\right)\,a_{k,2}+1\left(\mathrm{mod}\,10\right)\ldots\ldots\ldots a_{k,i}+1\left(\mathrm{mod}\,10\right)$. In particular, $a_{kk} = a_{kk}+1\left(\mathrm{mod}\,10\right)$, which is a contradiction. Hence, there cannot be a bijection from $\mathbb{R}$ to $\mathbb{N}$.

Problems with set theory

Set theory is plainly inadequate as a theory of the arithmetical continuum

> **Conjecture**
>
> "Set theory is the foundation of mathematics. All mathematical concepts are defined in terms of the primitive notions of set and membership. In axiomatic set theory we formulate a few simple axioms about these primitive notions in an attempt to capture the basic "obviously true" set-theoretic principles. From such axioms, all known mathematics may be derived." Kenneth Kunen.[21]

The possibility exists in empirical science that one primitive may be derived from another. This is the goal of theory-building. Additionally, since empirical science addresses the question of what nature is separately from phenomenology, there is no obligation in science to employ a concept merely because it has a phenomenological origin. Notwithstanding this theoretical possibility, **the statement of Kunen above is palpably false** and rests upon multiple misinterpretations of both set theory itself and the scope of application of set theory.

1. The Axiom of Completeness is not a statement of first-order set theory. (By "set theory" Kunen definitely intends "first-order set theory".) That the classical, standard theory of the arithmetical continuum that is expressed in the work of Hardy and continues to be employed throughout mathematics at every level is not set theoretical is as plain as the end of one's nose. **Set theory without the Axiom of Completeness is completely unable to construct a single real number**, and hence it cannot be an adequate theory to express the arithmetical continuum.
2. There is also a misconception about set theory at work in this conjecture. **Set theory is not in fact founded upon the single primitive** of set membership. It has at least two other primitives. (a) The actual infinite, which is expressed in the axiom of infinity – there exists at least one limit level = there is a complete actual infinite collection. The complete collection of all finite numbers is denoted ω. **It is impossible by mere analysis of the concept of set membership to discover the concept of the actual infinite.** The relation is not analytic. Also embedded in set theory is a notion of the potential infinite. This is expressed by the symbol $<\omega$. **The potential and actual infinite are distinct concepts and it is impossible to derive the one from the other.**
3. It is usual to denote the set of natural numbers by $\mathbb{N} = \{0, 1, 2, 3, ...\}$. There is a systematic confusion within set-theory of the relationship between this set and the set of ordinals, ω. This is illustrated by the following statement of Wolff: "The members of ω are called *finite* ordinals or natural numbers. In other words, to a set theorist, $\omega = \mathbb{N}$."[22] This statement is palpably false. The set of natural numbers is a potentially infinite collection; whereas the set ω is an actually infinite collection. So they cannot be the same object.

---

[21] Kenneth Kunen [1980]

[22] Wolf, Robert S [2005] p. 82. This statement is encountered frequently: "In fact, the finite ordinals are the natural numbers" – Komjáth and Totik [2000] p.37. "It is a philosophical quibble whether the elements of ω are the *real* natural numbers (whatever that means). The important thing is that they satisfy the Peano Postulaes" Kenenth Kunen [1980] p.19. Levy, Azriel [2002] avoids the issue altogether *by never introducing* $\mathbb{N}$ *at all.*

4. **There is no primitive in set theory for extension**, and hence in set theory it is not possible to measure the arithmetical continuum! Any text book of set theory will contain a chapter on the real numbers.

   Definition, irrational number

   An *irrational number* is a non-void proper initial segment d of $\mathbb{Q}$ … such that d has not greatest member and $\mathbb{Q} \sim d$ has no least member. [23]

   This *defines* what an irrational number is *but it does not construct it*, nor does it say that it exists. This definition mirrors exactly the Dedekind Completeness Axiom, which may be expressed succinctly by: "Any non-empty subset of $\mathbb{R}$ which is bounded above has a least upper bound in the set. The least upper bound is called its *supremum*." But the full expression of the Completeness Axiom requires second-order logic ("for all non-empty subsets…") and involves an existential statement (There exists a least upper bound…"). So this is the method of first-order set theory – **to convert existential constructions in second-order logic into definitions in first-order set theory**. The same principle is at work in the definition of the real line:

   Definition arithmetical continuum (Suslin)

   The real line (arithmetical continuum) is the unique separable, complete, linear order with neither end points nor isolated points.

   But how does a structure become separable and complete if it is not already so? **The existential construction of the real line (given by the Axiom of Completeness) is converted into a definition**. Granted that a structure is "separable, complete, linear…" then it is continuous and extended. But unless it is already extended that extension is not measured. Thus, in this way, **the other primitives needed for the arithmetical continuum are concealed within subtly constructed first-order definitions in set theory.** It is possible in first-order set theory to define an extension, but it is not possible to construct one, if it is not already given.

The nature of the proof that may be expected is an exercise in philosophical mathematics

The Continuum Hypothesis is formulated in the language of first-order set theory – a rigorous theory to be sure – with axioms and rules of inference that derive from the rules of first-order logic in which it is embedded. However, this language is inadequate to construct the arithmetical continuum of classical mathematics, which is constructed by the Axiom of Completeness in a different language – the language of second-order concepts – whose logic is without formal rules and is akin to natural language. So the character of a proof of the continuum hypothesis would take the form in part of a translation between the two languages. Much of it will be a work demonstrating that the meaning (philosophical term "intension") of such-and-such a formulation in the one language entails or is equivalent to the meaning (intension) of such-and-such a formulation in the other, where here the term "equivalent" is an unanalysed notion common to both languages or a tool that belongs to the natural meta-language in which both are assessed. The reader must expect a different kind of proof or argument.[24]

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[23] Levy, Azriel [2002] p.219

[24] So the work is intensely philosophical – more to do with verbal reasoning and the interplay of concepts than with computation. Now in fact the idea that such proof or argument is strange in general and foreign in particular to mathematics is an illusion. This interplay of concepts that

The inadequacy of set theory to represent analysis

One of the objections to first-order set theory concerns the potential and actual infinite and the confusion that has led to the statement $\omega = \mathbb{N}$. It turns out that refuting this statement is essential, and shall lead to some extraordinary observations about the adequacy of set theory to represent classical analysis.

The set of all natural numbers

In number theory we have the natural numbers, 1, 2, 3, … and we aggregate these into a collection that we call "the set of all natural numbers"; denoted $\mathbb{N}$. It is a defining property of this aggregate that it does not have a largest member, a supremum, and this is the Archimedean property.

> **Archimedean property**
>
> If a and b are any particular integers, then there exists a positive integer n such that $na > b$. (Burton [1976] p.2) This implies that $\mathbb{N}$ is not bounded above.

While this is a result of number theory, it is also possible to derive it from the Axiom of Completeness.[25]

The actual infinite in first-order set theory

The treatment is exemplified by the following result in set-theory:

> **Lemma, (Potter)[26]**
>
> For ω the following statements are equivalent: -
>
> 1. ω is a limit ordinal
> 2. $(\forall n)(n < \omega \supset n+1 < \omega)$
> 3. $\omega = \sup_{n<\omega} n$

On the assumption that $\omega = \mathbb{N}$, this is a manifest contradiction with number theory and thereby analysis based on the Axiom of Completeness.

Distinctions between the theories

Set theory is a theory of actually infinite aggregates. Number theory is a theory of the potentially infinite aggregate of all natural numbers. Analysis is a theory of both types of aggregate. If we take set theory on its own, that is, without reference to analysis, then it is not essential to distinguish within it between the potential and actual infinite. All collections are treated as complete collections,

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involves the assessment of the adequacy of our model building both for our subjective experience, our phenomenology, and for our hypothesis building in empirical science is, and always has been, the essence of philosophical science. It is only a contemporary misconception that asks us to imagine that our formulae and their rules stand alone, apart from the questioning mind that fashions them.

[25] **Analytic proof of the Archimedean property from the Completeness Axiom.** Suppose $\mathbb{N}$ is bounded above. Then by the completeness axiom there exists a unique real number u, such that $u = \sup \mathbb{N}$. For any number $n \in \mathbb{N}$ the number $n+1 \in \mathbb{N}$, hence $n+1 \le u$ and $n \le u-1$. This is true for all $n \in \mathbb{N}$, hence $u-1$ is an upper bound for $\mathbb{N}$. This contradicts the uniqueness of u, so $\mathbb{N}$ cannot be bounded above. (It might be objected that in set theory, the unique real number that is the supremum of $\mathbb{N}$ is $\omega = \sup \mathbb{N}$ and that the expression, $\omega - 1$, is meaningless. However, in the argument above u is a real number, so we may subtract from it. It is not assumed that the supremum is ω.

[26] The proof may be found in Potter [2004] p. 181. I have slightly adapted the theorem in Potter which is for all limit ordinals. Only ω concerns us here.

notwithstanding the occasional appearance of the symbol $<\omega$. But in analysis it is absolutely essential to distinguish between the potential and actual infinite, simply because expressions like $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ use both notions simultaneously in the same expression. So we have in analysis the following operation:

Completion

The operation of taking a potentially infinite collection and presuming it to form a completed totality; the operation of forming an actually infinite set from one that is potentially infinite.

Completion is conceptually presupposed by the Axiom of Completeness where it is equivalent to forming a limit from a potentially infinite convergent sequence. In analysis we have;

$\mathbb{N}$ The potentially infinite aggregate of natural numbers

ω The actually infinite aggregate (set) of all finite ordinals.

Let us analyse this further. In set theory we do see the symbol $<\omega$ which frequently does the work of the set of natural numbers, $\mathbb{N}$. Compare the enumeration of $<\omega$ with that of ω:

ω $\{1, 2, 3, 4, ...\}$

$<\omega$ $\{1, 2, 3, 4, ...\}$

They appear to be the same collection, but they cannot be. The difference is expressed in the following distinction, which shall be crucial to our subsequent investigation.

$\mathbf{2}^{\omega}$ The actually infinite set of all binary expansions, with cardinality continuum.

$\mathbf{2}^{<\omega}$ The potentially infinite set of all binary expansions, with cardinality countably infinite.

This latter set is isomorphic to the set:

$\mathbf{fin} = \bigcup \text{all finite subsets of } \omega = \{\{1\},\{2\}, \dots ,\{1,2\}, \{1,3\}, \dots ,\{1,2,3\}, \dots \}$.

It has only finite subsets, but the subsets of $\mathbf{2}^{\omega}$ has all kinds of infinite cosets of which $\omega - \{1\}$, $\omega - \{2\}$, $\omega - \{1,2\}$, $\omega - \{\text{all even numbers}\}$ are examples. In $\mathbf{2}^{\omega}$ there is a boundary between the finite subsets and the infinite cosets. The difference between the two sets is so profound that it may be said to express the hidden secret key to unlock the mystery of the classical concept of the continuum, as the remainder of this paper shall strive to clarify. The same distinction crops up with the distinction between cardinal and ordinal exponentiation:

Ordinal exponentiation $\underbrace{2\cdot 2\cdot 2\cdot \dots}_{<\omega \text{ times}}$ exponentiation by potential infinity

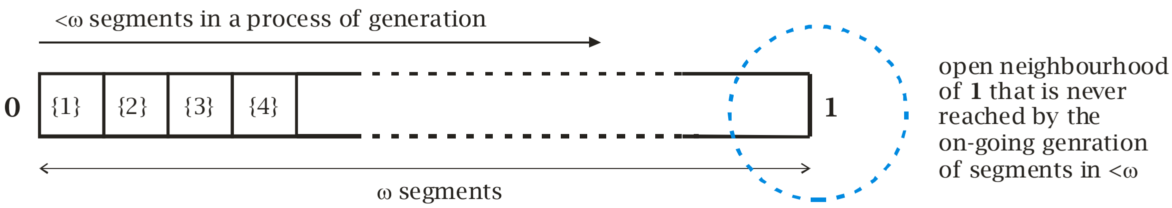
Cardinal exponentiation $2^{\aleph_0} = \underbrace{2\cdot 2\cdot 2\cdot \dots}_{\omega \text{ times}}$ exponentiation by actual infinity

The size of the former set is countably infinite, but cardinal exponentiation generates continuum many elements. I would recommend that we write $\mathbf{2}^{<\omega}$ for ordinal exponentiation, which otherwise appears to have no distinct notation, being understood only from context. Since the relation between the pairs $\omega, \mathbf{2}^{\omega}$ and $<\omega, \mathbf{2}^{<\omega}$ is one of exponentiation, this has an inverse, its logarithm. That is:

$$\omega \underset{\text{log}}{\overset{\text{exp}}{\rightleftarrows}} \mathbf{2}^{\omega}$$

$$<\omega \underset{\text{log}}{\overset{\text{exp}}{\rightleftarrows}} \mathbf{2}^{<\omega}$$

Hence, by comparing the logarithm of 2^{ω} with $2^{<\omega}$ we see that since there is in 2^{ω} an inexhaustible boundary between the infinite cosets and the finite subsets, which does not exist in $2^{<\omega}$ then there must also be a boundary within ω so that $<\omega$ represents a proper part of ω that is in potential ongoing generation but never actually generates the whole of ω. This proper part $<\omega$ has an indeterminate "boundary" with the part of ω that it can only potentially reach, but never actually reaches. So the relation of containment – that $<\omega$ is a proper part of ω – cannot be expressed in the notation of first-order set theory. To illustrate this. Let us suppose that we can partition the unit interval into actually ω segments.



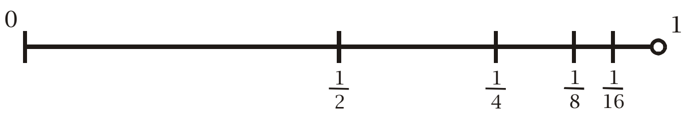
As in the diagram, suppose we start at **0** and start counting the segments up.[27] Then we can never reach **1** in any finite number of steps. That means that there is a neighbourhood of **1** that the potentially infinite process of generating segments never reaches into. So there is an indeterminate boundary, represented in the diagram by the dashed blue line between the potentially infinite part of the partition and the actually infinite part. Hence ω is a larger aggregate than $<\omega$.

The aggregate of all rational numbers

We have just seen that in analysis certain infinite aggregates exists in two pairs – one potentially infinite and expressing the ongoing and never completed generation of a sequence; the other actually infinite and expressing the notion of that process of generation actually completed. Thus, for example, is the set of all rational numbers, $\mathbb{Q}$, a potential or an actual infinity? To answer this, I should say that $\mathbb{Q}$, on analogy with $\mathbb{N}$, is a potential infinity, *and there is a need for another symbol to denote the set of rationals when thought of as completed and actually infinite aggregate.* The set of rational numbers is said to have order-type η, which is the order of the rationals with the natural order relation $<$. Under this relation the expression $\frac{a_1}{b_1} < \frac{a_2}{b_2}$ means that between $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ there is always another rational $\frac{a_1}{b_1} < \frac{p}{q} < \frac{a_2}{b_2}$ and the set $\mathbb{Q}$ has neither first nor last member. This is what the order type η means. It thus makes sense to use the symbol η for the actually infinite aggregate of rational numbers where $\mathbb{Q}$ is the potentially infinite aggregate.

$\mathbb{Q},<$	Potentially infinite aggregate of rational numbers subject to the natural order
$\eta,<$	Actually infinite aggregate of rational numbers as a distinct order-type.

[27] Zeno's Dichotomy



shows that this is coherent.

But this is relative to the order relation, for by the Cantor diagonalisation process the set of rationals can be well-ordered in order of generation. The ability to order the rational numbers in this way by generation also lies at the heart of the subject: $\mathbb{Q}$ by order of generation is isomorphic to $\mathbb{N}$ ordered by successive numbers; the order type of $\mathbb{Q}$ under this order is ω.

Certain objects have simultaneously different descriptions with different properties

Hence we uncover another vital fundamental principle at work: that the same aggregate can have different descriptions, and *that this aggregate is the object that has both those descriptions.* The set $\mathbb{N}$ just is the set of natural numbers, and to all intents and purposes it has just this one description. But the set $\mathbb{Q}$ is **simultaneously**

1. A potentially infinite aggregate subject to the natural order whose order-type upon completion is η.
2. A potentially infinite aggregate subject to the process of generation[28], whose order-type upon completion is ω.

The aggregate of all dyadic fractions

Another important fact is that $\mathbb{Q}$ by order of generation is isomorphic to the aggregate of dyadic fractions, here denoted $\mathbb{D}$, which are fractions of the form $\frac{1}{2^n}, \frac{2}{2^n}, \ldots, \frac{n-1}{2^n}$ for finite $n \in \mathbb{N}$.

The dense subset of the continuum

It is a theorem that $\mathbb{R}$ is the closure of any dense subset of it. The standard dense subset is $\mathbb{Q}$, but there are others. In particularly, the set of dyadic fractions $\mathbb{D}$ is dense in $\mathbb{R}$. The nature of $\mathbb{D}$ depends also on how it is ordered, and like $\mathbb{Q}$ is simultaneously one object with two descriptions.

1. A potentially infinite aggregate subject to the natural order relation $\frac{a}{2^m} < \frac{b}{2^n}$ iff $\frac{2^n a}{2^{m+n}} < \frac{2^m b}{2^{m+n}}$, with order type η.
2. A potentially infinite aggregate subject to the process of generation, whose order type is $\mathbf{2}^{<\omega} \cong \omega$.

In practice in this context we usually think of $\mathbb{D}$ in order of generation, and have to recall on those occasions where it is needed that it is also a dense subset for $\mathbb{R}$. The reason its order-type is best characterised as $\mathbf{2}^{<\omega}$ is that it may be identified with the potentially infinite binomial tree. The potentially infinite binomial tree, which we will denoted by $\mathbb{T}$, denotes *any potentially infinite dense subset of the continuum subject to the binomial order of generation*; it also has an actually infinite counterpart, which in this case is also its closure.

> $\mathbb{T}$ is the potentially infinite binomial tree with order type $\mathbf{2}^{<\omega} \cong \omega$, whose completion and closure is the binary representing set for the real numbers, the Cantor set $\mathbf{2}^{\omega}$.

[28] The set of rational numbers can be generated by the diagonalisation argument cited above.

Summary

The operation of completing a potentially infinite into an actually infinite aggregate is always possible. It is no longer necessary to distinguish between cardinal and ordinal exponentiation for they are the same operation but depend on whether the aggregate is potentially or actually infinite. Thus ω is the completion of $\mathbb{N}$, and likewise

Potentially infinite aggregate	Actually infinite aggregate
$\mathbb{N}$	ω
$\mathbb{Q}$	η
$\mathbb{D} \cong \mathbb{T}$	$\mathbf{2}^{\omega}$
$\mathbf{Fin} = \mathbf{2}^{<\omega}$	M[29]

We can clarify other relationships as well:

Set	Exponent
$\mathbb{N} \cong < \omega$	$\mathbf{Fin} = \mathbf{2}^{<\omega}$
ω	$\mathbf{2}^{\omega}$

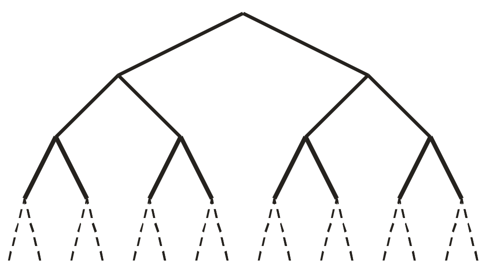
Set	Closure
η or $\mathbb{Q}$	$\mathbb{R}$
$\mathbb{D} \cong \mathbb{T}$	$\mathbb{R}$

The importance of the binomial tree is illustrated by this summary. It is a representing set for the dense subset of the continuum, so its closure is $\mathbb{R}$; its counterpart, the Cantor set, $\mathbf{2}^{\omega}$, is both its actually infinite aggregate and its exponent; and the Cantor set is a representing set for $\mathbb{R}$.

The structure of the arithmetical continuum under the classical conception and the Derived Set

The strange properties of the Cantor set and its even stranger subset, Fin

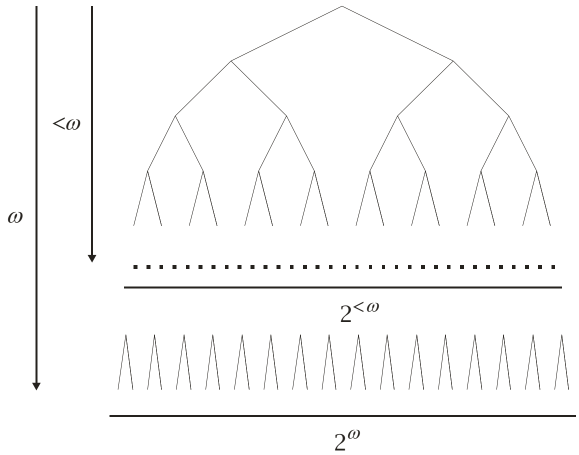
We need to make some further observations about the Cantor set. It is a tree structure with branches of actually ω length. However, as we generate this tree from repeated iterations of the branching process, we realise that we have a succession of finite trees embedded in the infinite structure.



This diagram illustrates the growth of the tree between the 3rd and 4th iterations of the power set operation: $\mathbf{2}^{n} = \{0,1\}^{n}$. This continues *ad finitum*. Thus, there is embedded in the Cantor set not only

[29] Here I introduce M to be used to denote the completion of $\mathbf{Fin} = \mathbf{2}^{<\omega}$.

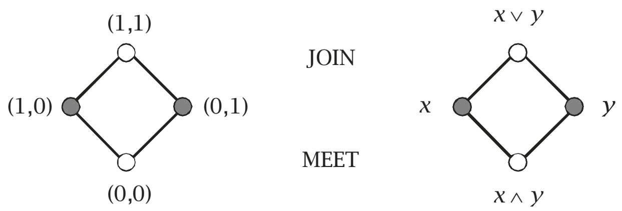
every finite tree $\mathbf{2}^n = \{0,1\}^n$ where $n \in \mathbb{N}$ but also the potentially infinite tree $\mathbb{T} = \mathbf{2}^{<\omega} = \{0,1\}^{<\omega}$. This a well-defined structure, but a strange one. In other words, the Cantor set is a very strange object.



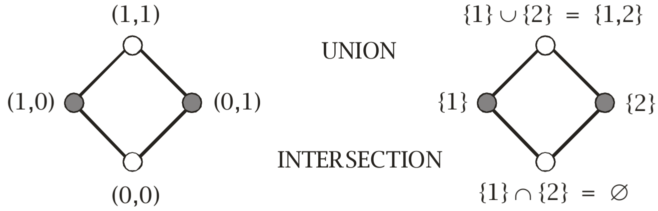
This diagram is for heuristic purposes only. Imagine that the last row in the diagram represents the final stage of taking the actually infinite branches of the tree. This is strictly impossible – but taking it on an "as if" basis, then before that final stage is reached there is still an infinite gap, and before that the finite tree is found, also in a process of infinite generation. The potentially infinite tree $\mathbb{T} \cong \mathbf{Fin} = \mathbf{2}^{<\omega}$ is embedded as a strict sub-set or sub-tree in the actually infinite tree $\mathbf{2}^{\omega}$.

Lattice structure

We observed above that for the finite case each tree is a vector space. The members of each finite tree are n-tuples of ordered sets. For example, for $n = 2$ we have, $\mathbf{2}^2 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$. Now since $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we have a lattice structure:



(Row vectors replace the column vectors in the diagram for convenience.) So the set $\mathbf{2}^2$ is not only a tree and a vector space, but also a Boolean algebra, which is a complete, distributive lattice. This applies also to the actually infinite case $\mathbf{2}^{\omega}$ as well. We can take the matter further. Suppose we label sets above by $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \{1\}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \{2\}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \varnothing, \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \{1,2\}$, this displays the structure as a field of sets – that is, as the result of taking unions and intersections of sets. Note, the objects $\{1\},\{2\}$ are labels only, and no meaning should be attached to the '1' and '2' used; we could have used other symbols: $\{a\}, \{b\}, \{a,b\}, \varnothing$.



The vectors labelled $\begin{pmatrix}1\\0\end{pmatrix}=\{1\}$, $\begin{pmatrix}0\\1\end{pmatrix}=\{2\}$ are called atoms, and any finite Boolean algebra, $\mathbf{2}^n=\{0,1\}^n$, has a set of n atoms. It is said to be *atomic*. The defining property of any two atoms is that they have empty intersection. The Cantor set, $\mathbf{2}^\omega=\{0,1\}^\omega$, is also atomic with ω atoms. For $n=2$ the set $\{\varnothing,\{1\},\{2\},\{1,2\}\}$ is the power set of the set $\{1,2\}$ - the set of all subsets of $\{1,2\}$. This, of course, applies for any finite $n\in\mathbb{N}$; and likewise the Cantor set as a whole is the set of all subsets of $\omega=\{1, 2, 3, \ldots\}$ where, of course, the set is conceived of as an actually infinite, complete totality.

Fin

Now what of the potentially infinite tree that is embedded within the Cantor set; the object denoted by $\mathbb{T}\cong\mathbf{2}^{<\omega}=\{0,1\}^{<\omega}$? This too may be regarded as the power set of a countably infinite set of labels, but this time not conceived as an actually infinite, completed totality, but as one that is potentially infinite. Thus $\mathbf{2}^{<\omega}=\mathbf{2}^{\mathbb{N}}$ is the set of all subsets of $<\omega=\mathbb{N}=\{1, 2, 3, \ldots\}$. In the literature this is designated by $\mathbf{Fin}=\mathbf{2}^{<\omega}=\bigcup \text{finite subsets of } \omega$. The properties of **Fin** are very strange, and it is very difficult to "get one's head around them"; furthermore, the embedding of **Fin** within the Cantor set is also conceptually challenging. The main point is that **Fin** is non-atomic. **Fin** may be thought of as a structure that is always in the process of on-going generation; if it were an object in time we would see it as a perpetually never ending video of the Cantor tree growing and growing and growing. The Cantor set represents the (for us) "impossible" notion of that video coming to an end, after an actually infinite duration of time. For this reason **Fin** is not a Boolean algebra, because it is never possible to take an empty intersection of its members – the null set is never obtained. If we imagine **Fin** as in the video as this ever-growing tree, then we can also imagine freezing the video at some point and pretending that we have generated a set of atoms; but then the video starts up again and these imaginary "notional" atoms turn out after all to have a further non-empty intersection. So **Fin** is always in a process of on-going generation.

The Cantor set is not the arithmetical continuum, and as it is the only possible candidate for the arithmetical continuum within set-theory, set-theory could not possibly be a theory of the continuum

It is high-time that we dealt with the lingering possibility that the Cantor set is, after all, the arithmetical continuum. For, after all, the continuum hypothesis, expressed in ordinals, is $\mathbf{2}^\omega=\omega_1$ and $\mathbf{2}^\omega$ is the Cantor set. Here I do not go into the construction of the Smith-Votara-Cantor (SVC) sets, which is a fascinating story, but just cite the outcome of that work, which is to demonstrate that the Cantor set is meagre.

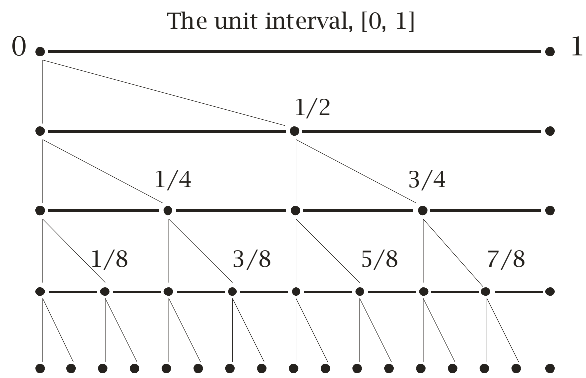
Definition, meagre

A set is said to be *first category* or *meagre* if it can be represented as a countable union of nowhere dense sets.[30]

The result of work on the SVC sets shows that the Cantor set is meagre. This means that the Cantor set is perfect (equal to its own closure), non-continuous, nowhere dense (totally disconnected) and has outer content (measure) equal to zero. Since the unit interval $[0,1]$ is perfect, continuous, has a dense subset and has unit measure, the Cantor set is not homeomorphic to it. Thus, the Cantor set is not the arithmetic continuum.

But this should not be such a surprise for when we fill up a line with a Cantor set of points of zero measure we always leave the extension of the line behind

But this should not surprise us. The whole underlying challenge is to construct the continuum out of its points. This is part of the intuition on which our notion of the continuum is founded; or, to repeat G.H. Hardy again: "...It is convenient to suppose that the straight line ... is composed of points corresponding to all the numbers of the arithmetical continuum, and of no others." One of the fundamental intuitions missing from the set-theoretical approach is that **one must incorporate extension as a primitive notion**. We have already seen how this need has been obscured by the strategy within set-theory to replace second-order existential statements by first-order definitions. In actual fact, we can never generate an extension, for extension is primitive and must be just given; but what we can do is display its synthetic relation to the points from which it is made, and this is what the Completeness Axiom achieves. So now suppose that we start systematically to remove points from an extended portion of the line – let us say the unit interval $[0,1]$.



Let us suppose that each of these points, or boundaries, has absolutely no extension – hence, of measure zero. Then at no stage of the process has the extension of the interval $[0,1]$ been diminished. Now even an actually infinite ω number of iterations of this process will not generate a measure. So the Cantor set in such a case merely perforates the extended interval without ever filling it up. For this reason the Cantor set is not the arithmetical continuum.

The arithmetical continuum is a double tree

So we start with a neighbourhood of 1, $(0,1]$, a primitive extended open set.

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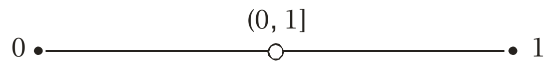
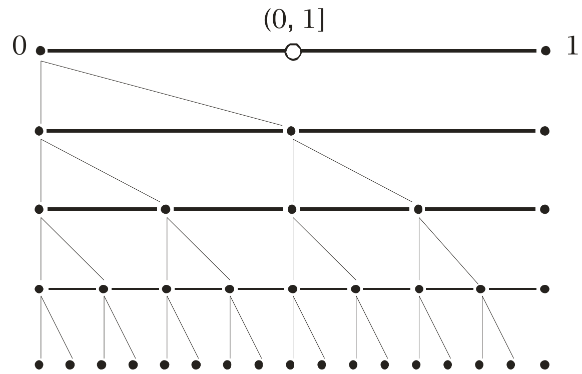
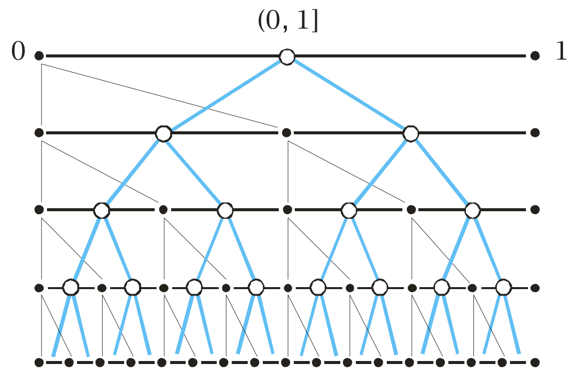
[30] The subject of what a meagre set is and a definition of "nowhere dense" is clarified in the subsequent discussion of the null-meagre decomposition of the continuum. See below.



In this diagram the small black dots represent points of zero measure, whereas the open white circle stands for the content of an interval – a primitive portion of extended space, existing either in transcendental reality independent of consciousness, or presented to consciousness –however the reader wishes to think of it. Note, in this case $(0,1]$ also contains the point labelled 1. Into this neighbourhood we start to systematically introduce points by doubling and doubling their number.



This generates one binomial tree $\mathbb{T}$ whose actually infinite completion is the Cantor set. But as we do so we also divide and divide the open extended neighbourhood of $(0,1]$:



So we generate two trees not one – and on completion, taking the branches to actually ω iterations, two Cantor sets, not one. In other words, the reason why a single Cantor set is not the arithmetical continuum is because the arithmetical continuum needs to be represented by two interlocking Cantor sets, not one. In one respect it has the character $\mathbf{2}^{\omega} \oplus \mathbf{2}^{\omega}$, where $\oplus$ represents some kind of union of two sets, though a union that is extraneous to the theory of sets, and cannot be represented in that language. I shall clarify this as we proceed.

* When introducing boundary points into the continuum, the obvious choice is to represent these by $\mathbb{Q} \cap [0,1]$, which is the canonical countable dense subset of the unit interval; however, in this context it is preferable to use the set of dyadic fractions $\mathbb{D} = \bigcup_{n \in \mathbb{N}} \left\{ 0, \frac{1}{2^n}, \frac{2^0 + 2^1}{2^n}, \ldots, \frac{\sum_{r=0}^{n-1} 2^r}{2^n} \right\}$ because it can be generated as a binomial tree structure $\mathbb{T}$. A theorem of Cantor shows that any two countable dense subsets of $\mathbb{R}$ are isomorphic. Thus $\mathbb{Q} \cong \mathbb{T} \cong \mathbb{D}$ and we may regard $\mathbb{D}$ as a tree representation of $\mathbb{Q}$.

Keeping track: how are we doing for proof? Where are we going?

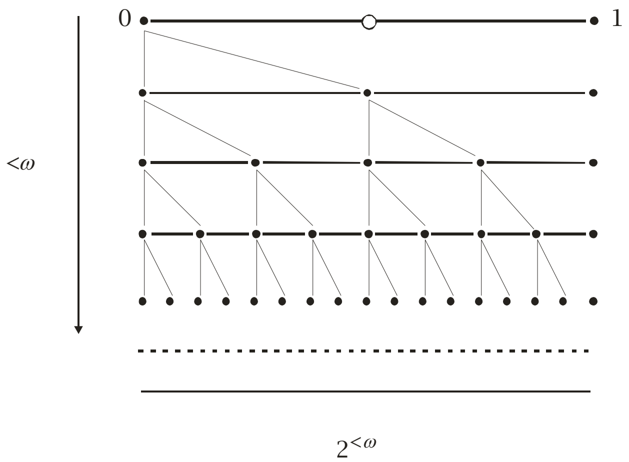
The goal of mathematicians is to prove something, though "What is proof?" is more often than not an unanalysed notion even in mathematics. I've already warned the reader not to expect an argument couched in the language of the axiomatic method. It seems heuristically better to present the model first and subsequently to demonstrate that it is the one and only model generated by the Axiom of Completeness. But it may be as well to forestall inevitable criticism by doing a quick stock-taking now. Thus, what do we know about the arithmetical continuum?

1. It is generated from a countable dense subset – the set of all rational numbers. The Axiom of Completeness not only states this, but tells us how it is generated.
2. In the language of set-theory, the Cantor set is not the arithmetical continuum, though it has as many objects as there are in the continuum and may represent it. Furthermore, the thing missing from the Cantor representation is precisely the primitive and unanalysable notion of extension – a continuum that takes up space, and is space.

Taking the first point then we see that we must extract from the continuum a countable dense subset, at the same time identifying those with points of the continuum. That is precisely what we have done with the first of our tree structures. The second point forces us to add the one missing ingredient – the missing extension that the removing of the countable dense subset can never exhaust. **So we must be on the right lines**. The main thing to demonstrate is that the countable dense subset that has been removed does indeed, upon closure, generate the whole continuum. After that we need to bring the whole structure into harmony with the language of set-theory and reflect upon the interaction of the model with that theory. Then we should check that the known results concerning the continuum are consistent with the model, so that there are no lurking doubts. Finally, we should see whether we can in fact on the basis of the model answer the question posed by the Continuum Hypothesis as to the size of the continuum. So that is the outline of the work to come.

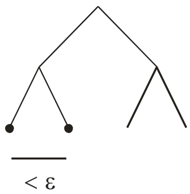
The process of generation, and two kinds of end point

Let me draw out some of the implications of the two trees model first. The first tree:



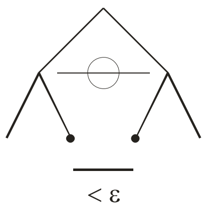
generates the countable dense subset, $\mathbb{T} \cong \mathbb{Q}$ in the arithmetical continuum, and does so in a way analogous to the Cantor diagonalisation argument demonstrating that $\mathbb{Q}$ and $\mathbb{N}$ have the same cardinality. It is a potentially infinite structure whose height is $<\omega$ and whose width is $2^{<\omega}$. The end points are in perpetual generation, and any "snapshot" of them reveals a finite set, that we equate with a set of rational numbers, never the whole set. In the limit, by taking branches that are actually ω in length, we reach down and finally generate the whole set. But, of course, we do much more – for we at the same time generate all the real numbers. There is something of a paradox here. The potentially infinite tree shown in the diagram above generates the dense subset of the continuum, $\mathbb{Q}$, but it does so in stages. If we regard this stage-by-stage process as complete and close the set $\mathbb{Q}$ we have to *at the same time* generate the whole of $\mathbb{R}$; it is not possible to separate the completion of one from the other. (This is what is meant by saying that the closure of $\mathbb{Q}$ is $\mathbb{R}$, or equivalently $\mathbb{Q}$ is not a perfect set and not equal to its own derivative set.) So what kind of point is generated at the closure? The first thing to observe is that there will be different kinds of end-point.

(1) Let's start by looking at the first kind.



This diagram operates under the convenient fiction that we can visualise the last two nodes of the infinitely long chain. But what this is saying is that the gap between the two adjacent branches is so small that we are unable to distinguish them by any given measure; since there is a rational number smaller than any given real number, and vice-versa, what we are saying is that this gap between the two adjacent branches is so small that we cannot measure it by any given rational number. Thus, the notion of branches that are actually infinite in length is no much more than a façon-de-parler for this situation, from whence it derives its inner coherence as a concept. This tree that we have generated comprises points of zero measure. In the limit we are saying that the distance between the two adjacent branches is so small that *to all intents and purposes* the two points are *one and the same point.* (This is the fundamental notion behind a limit.) Since both points are equated, and since both are points of absolutely zero measure, we must also infer that the identified point has zero measure. There will be $\mathfrak{c} = 2^{\aleph_0}$ of these.

(2) So our first kind of end point, our first kind of real number, has zero measure – it's just a point in the usual sense of point. Consequently, using these alone, we cannot fill up the line, an infinite number of points of zero measure will not create an interval – the extension has still not been generated. So we need another kind of point, and the tree diagram shows where this point is generated.



The space between adjacent branches is just that – a space, or an extension. These gaps between the branches originate from the infinite subdivision of the neighbourhood of 1 that is $(0,1]$. This was the primitive extension. The extension of the line has been subdivided infinitely, but it has never been exhausted. So in the limit the pieces between the branches are extended portion of space with a size that is **not zero but incommensurable with zero**. Hence they are **infinitesimals**, in the sense of Leibniz. This follows a principle that we propose as an Axiom governing the primitive notion of extension:

> **Axiom of Indestructibility of Extension**
>
> Every proper part of an extended portion of space is extended.

Or in the words of Kant:

> The property of magnitudes by which no part of them is the smallest possible, that is, by which no part is simple, is called their continuity. Space and time are quanta continua, because no part of them can be given save as enclosed between limits (points or instants), and therefore only in such fashion that this part is itself again a space and a time. **Space therefore consists solely of spaces, time solely of times**. Points and instants are only limits, that is, mere positions which limit space and time.[31]

There are limits called **infinitesimals** that have measure incommensurable with zero, and are still extended.[32]

(3) I observe that the first kind of point mentioned here also arises from the squeezing together of an open interval. So, given the Axiom of Indestructibility of Extension this point is also an extension. However, the distinction between the two kinds of point is that the first kind is that kind of extension that may be called a "point of zero measure" because a continuum many collection of them still has zero measure. But for the second point, which we are calling an "infinitesimal" – continuum many of them does amount to a measure. The size of the measure is indeterminate – which arises from the fact that there are continuum many points in any interval.

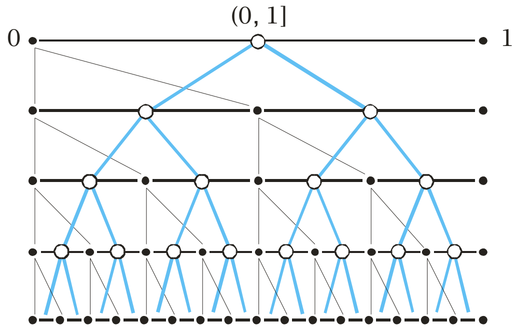
(4) However, the black dots in the diagram represent boundaries introduced by the process of generating the binomial tree. Hence, they can be regarded as points of absolutely zero measure. They are just points because they never were extensions or derived from an interval by taking a limit. So there are three kinds of point in the arithmetical continuum: boundaries, points of zero measure and infinitesimals.

Factoring the double tree into a single tree

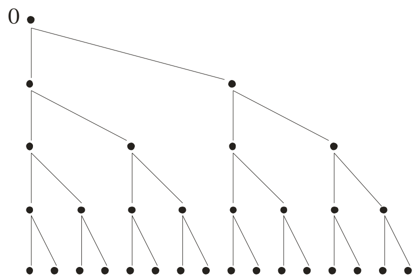
While we cannot say how the points and infinitesimals relate to each other on the continuum, the situation is clearer in the case of the potentially infinite tree that generates them.

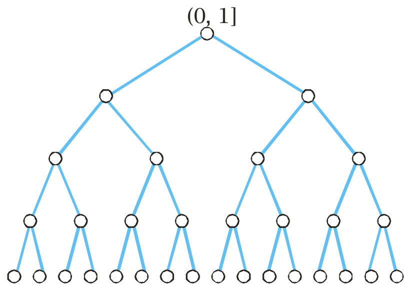
[31] Kant, Immanuel [1982 / 1781] Critique of Pure Reason. Part 2. Transcendental Logic. First Division. Chapter II. § 3.2 Anticipations of Perception. Translated by Norman Kemp Smith, First edition 1929. (Reprinted 1982) p. 204. My highlighting.

[32] The diagram suggests that points and extensions alternate in the real line. However, the process of actually completing the Cantor set is underdetermined even in this model. Some of the end points must correspond to rational numbers and how those juggle with the other reals is not determined. So we cannot say absolutely how the real line would look "under a microscope" so to speak and what one point being adjacent to another might mean. It is sufficient to know that on the real line there are two kinds of points – points of zero measure and infinitesimals.



Here the two trees are paired off and at each stage there is for every one point and additional division of the interval The sum total of the points are the dense subset that punctures the unit interval, and sum total of the intervals is the neighbourhood of 1 less hat dense subset.

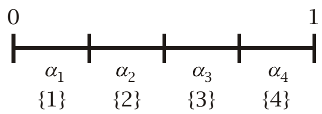




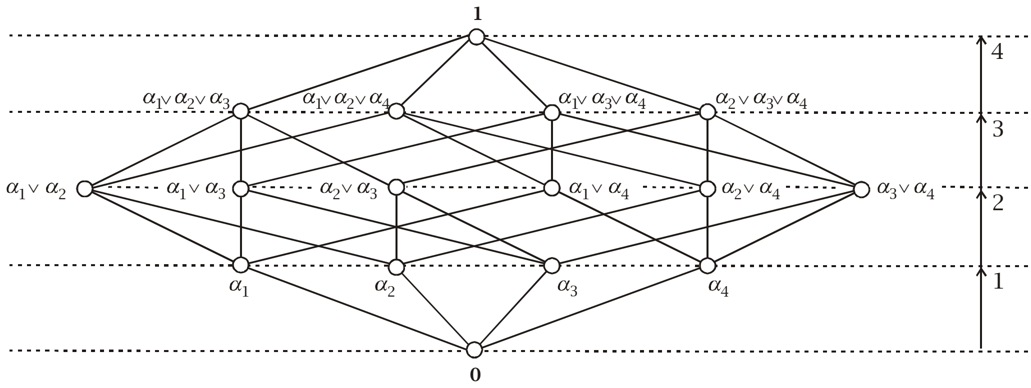
These are our two structures: $\mathbf{2}^{\omega} \oplus \mathbf{2}^{\omega}$.

Exponential and logarithm upon a lattice – the skeleton of the continuum

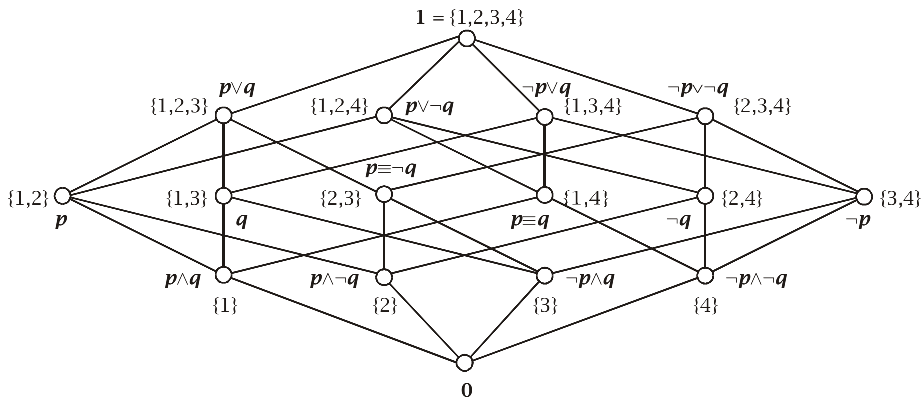
As a concrete illustration, suppose we divide up the unit interval into four pieces that we shall call atoms.



In this, the pieces (atoms) are mutually exclusive intervals that together add up to the whole interval. We could iterate the process of dividing up the line into pieces by subdividing these given pieces *ad infinintum*. For any given finite stage of this process the set of atoms that partition the real line is called the **skeleton** of the line. By taking all lattice joins or equivalently set unions of these atoms (pieces) we obtain a finite Boolean algebra. For example, suppose for stage $n = 2$ we have $\mathbf{2}^2 = 4$ atoms in the skeleton as in the diagram above. We may label the pieces $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ but it is also convenient to use numerical labels: $\{\{1\},\{2\},\{3\},\{4\}\}$. Then by taking lattice joins (set unions) of these we generate the Boolean algebra that may diagrammatically be represented as follows:

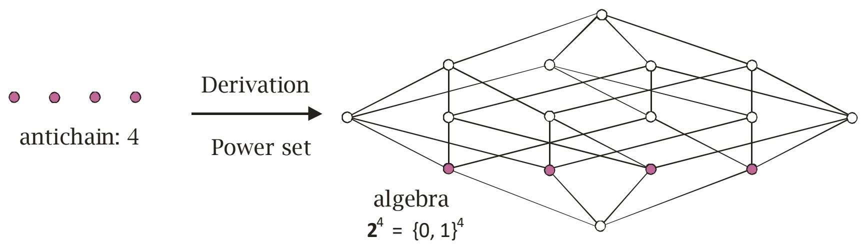


This displays the lattice with lattice joins, and also as a metric space with the distance function shown on the right. It is equivalent to a field of sets[33] with diagram:[34]



The exact derivation

The relationship between the skeleton and the lattice is an *exact derivation*. **From the skeleton the lattice as a whole may be constructed, and conversely, given a lattice only one skeleton is determined.** The derivation is by means of ordinal exponentiation, or equivalently by means of the power set operation to generate the lattice elements as set of all sub-sets of the atoms. With regard to the continuum this tells us that the continuum has a lattice structure, and is generated from its skeleton. *The task is to find the skeleton of atoms that generates the entire continuum.* The skeleton comprising a set of atoms is also called an **anti-chain.**



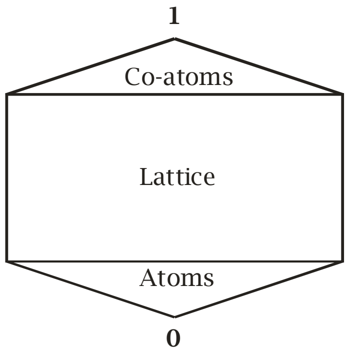
The atoms are just one basis for the Boolean algebra. We can also generate the whole lattice from the **co-atoms**. In the algebra above this is the set:

[33] This is the content of Stone Representation Theorem, which extends to the case of an infinite lattice, though subject to the Axiom of Choice.

[34] This diagram also displays an application of this lattice to first-order propositional logic with the rule that all valid inferences proceed up the lattice. (For instance, $p \wedge q \vdash p \vdash p \vee q$.) For the skeleton of $2^2 = 4$ atoms, the lattice has $2^{2^2} = 4^2 = 16$ nodes. The topmost node is denoted **1** and represents the entire space. The bottom node is denoted **0** and represents the null space. In the application to information **1** stands for necessity and **0** for impossibility.

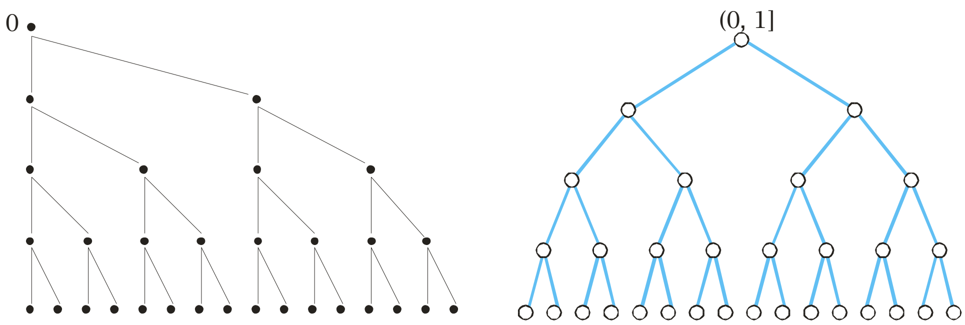
$$\{\mathbf{1}-\{1\},\mathbf{1}-\{2\},\mathbf{1}-\{3\},\mathbf{1}-\{4\}\}=\{\{2,3,4\},\{1,3,4\},\{1,2,4\},\{1,2,3\}\}.$$

We have the following schematic diagram of any Boolean algebra.

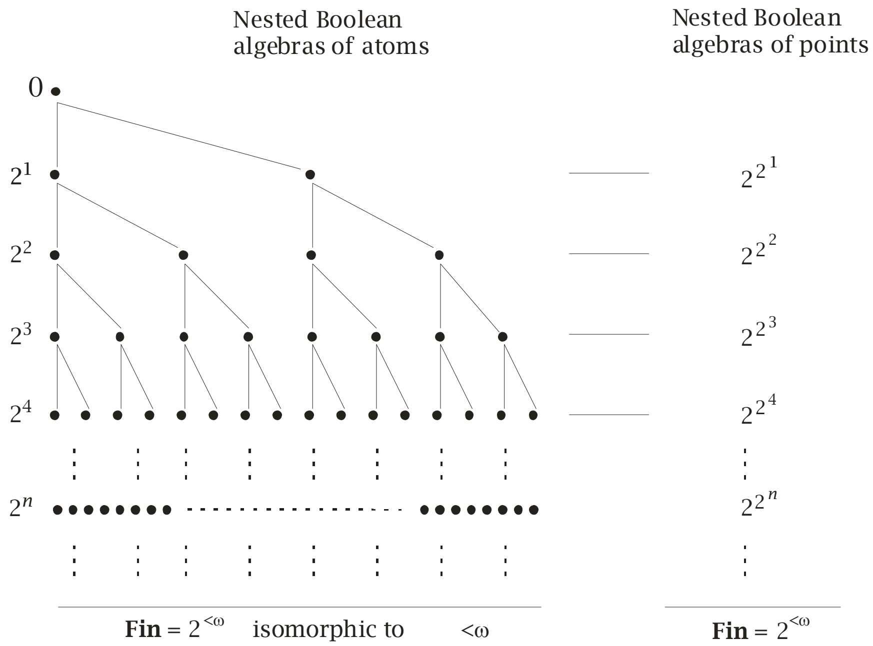


Boolean lattice version of the two trees model

Our two trees model has all of the boundary points on the left and all of the intervals comprising the neighbourhood of 1 on the right.



Both $\mathbb{T}$ trees are in a process of ongoing generation as their branches reach down and down in a never terminating potential infinity. Each of these trees is isomorphic to $\mathbf{Fin}=\mathbf{2}^{<\omega}=\bigcup$finite subsets of ω. This means the cardinality of both is $\aleph_0$. The left hand tree is a collection of points corresponding under a different description to rational numbers in the interval $\mathbb{Q}\cap[0,1]$, so this is a collection of atoms of a Boolean algebra derived by exponentiation. In other words all these atoms correspond to points of zero measure, and as there is only a potentially infinite number of them and space has $2^{\aleph_0}$ points, the collection is incomplete. In the right-hand tree every node is an interval, and what is happening at each level is that the intervals are being sub-divided further and further in a refinement of the partition of the unit interval $[0,1]$. So none of these at any finite stage correspond to a point. Regarding the left hand tree of atoms, this generates via exponentiation a corresponding further Boolean algebra of points.



On the left of this diagram we are systematically puncturing the unit interval with points that we associate under a different description with rational numbers of $\mathbb{Q}\cap[0,1]$. As we do so we generate another aggregate whose size is shown on the right, the Boolean algebra of which these are atoms. The elements of this Boolean algebra we can associate with the algebraic numbers generated by the rational numbers corresponding to the atoms. These algebraic numbers also correspond to points of the continuum.

Representation by a countable set of labels

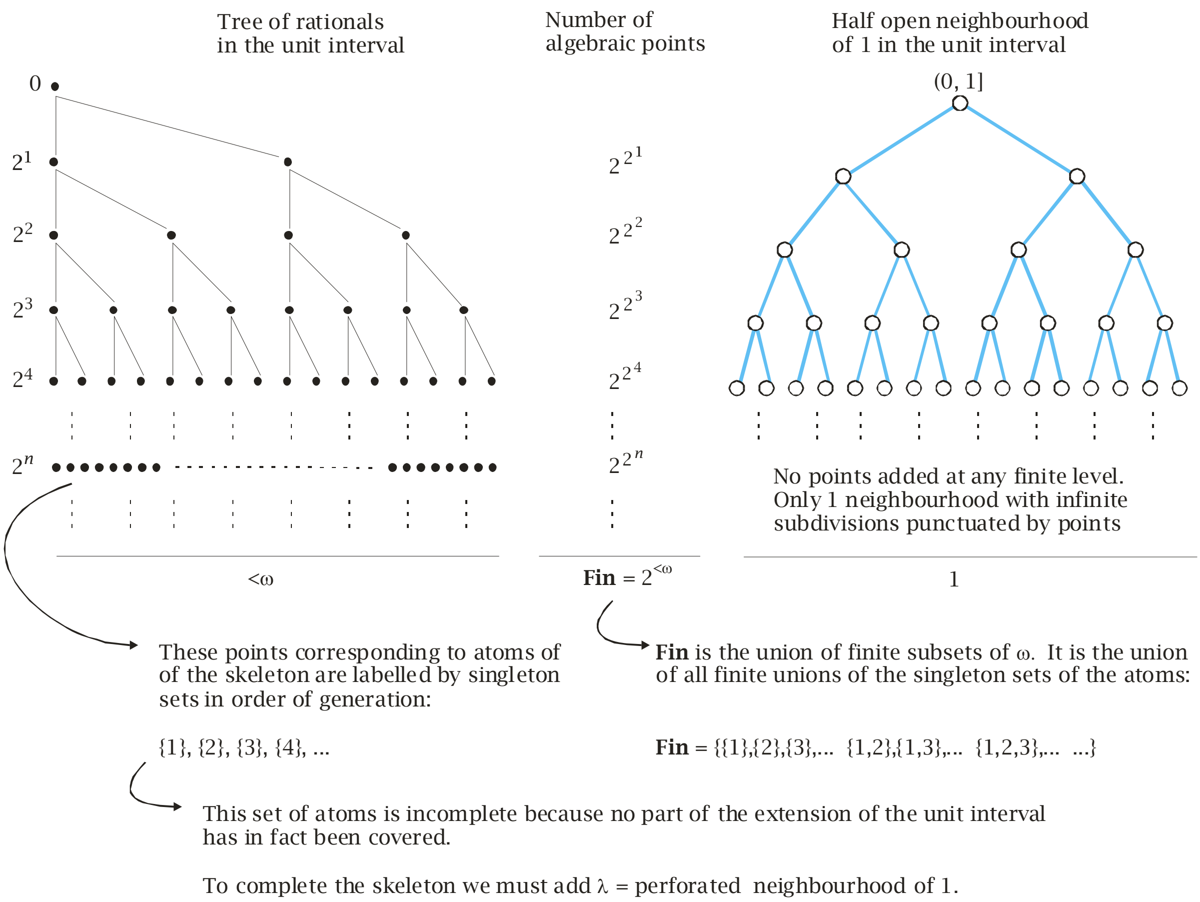
We use Cantor diagonalisation to establish a correspondence between the members of $\mathbb{Q}\cap[0,1]$ and the members of $\mathbb{N}$. In order to avoid duplication (since an atom cannot be duplicated), we delete from the diagnolisation procedure all repeated instances of an equivalent fraction, treating the fractions themselves as equivalence classes and using the fraction in its lowest cancelled down form as a representative.

$$\begin{array}{ccc} \mathbb{Q} & \longrightarrow & \mathbb{N} \\ 0 & \longrightarrow & 1 \\ \frac{1}{2} & \longrightarrow & 2 \\ \frac{1}{3} & \longrightarrow & 3 \\ \frac{2}{3} & \longrightarrow & 4 \\ \frac{1}{4} & \longrightarrow & 5 \\ \frac{1}{5} & \longrightarrow & 6 \\ \ldots & \longrightarrow & \ldots \end{array}$$

So we assign to every rational point in $\mathbb{T}$ identified under a different description with $\mathbb{Q}\cap[0,1]$ a natural number. But we wish to demonstrate the construction of the Boolean algebra by exponentiation as an algebra of sets, so it is preferable to use singleton sets for these labels.

$$
\begin{array}{ccc}
\mathbb{Q} & \longrightarrow & \mathbb{N} \\
0 & \longrightarrow & \{1\} \\
\frac{1}{2} & \longrightarrow & \{2\} \\
\frac{1}{3} & \longrightarrow & \{3\} \\
\frac{2}{3} & \longrightarrow & \{4\} \\
\frac{1}{4} & \longrightarrow & \{5\} \\
\frac{1}{5} & \longrightarrow & \{6\} \\
\dots & \longrightarrow & \dots
\end{array}
$$

The set of these labels is denoted here $\mathbb{N}_{\mathbb{A}} = \{\{1\}, \{2\}, \{3\}, ...\}$. So referring back to our tree:



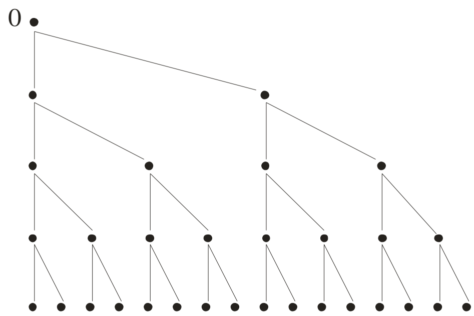
As the diagram explains we construct a skeleton of the unit interval as follows. Firstly, we punctuate the unit interval with a dense set $\mathbb{T}$ that we identify under a different description with the dense set $\mathbb{Q}$ with which we also pair off with a denumerable set of labels $\mathbb{N}_{\mathbb{A}} = \{\{1\}, \{2\}, \{3\}, ...\}$ to indicate that these are atoms of a Boolean algebra. This is not a complete skeleton of the unit interval, no part of the extension of the real line has been covered by it, for the measure of the set of rational numbers is zero. To complete the skeleton we should add back the half open interval $(0,1]$, which is the neighbourhood of 1. However, if we add back all of $(0,1]$ we will add the points of $\mathbb{Q}$ twice. We must add $\lambda = (0,1] - \mathbb{Q}$. According to the above, we associate $\mathbb{Q} \cap [0,1]$ with the set $\mathbb{N}_{\mathbb{A}}$ and we are labelling $\lambda = (0,1] - \mathbb{Q}$, so our model of the skeleton of the unit interval is: $\mathbb{N}_{\lambda} = \mathbb{N}_{\mathbb{A}} \cup \lambda$.

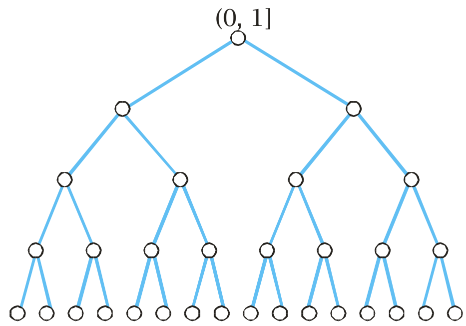
How many points are there in the skeleton? Combining the double tree into a single tree

The above argument shows that the skeleton of the unit interval is modeled by a structure represented by $\mathbb{N}_\lambda = \mathbb{N} \cup \lambda$. So the question arises: how many points are there in this structure? There are two possibilities: ω and $\omega + 1$.

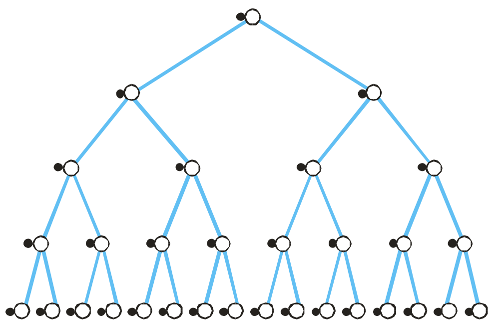
(1) To consider the latter possibility. Firstly, in order to obtain all the points of the real line we will have to close the rational dense subset; conceptually that means completing it into an actual infinity and then taking its power set. Here we are manifesting $\mathbb{Q}$ by order of generation, so its order type is ω and not η. By completing $\mathbb{Q} \sim \mathbb{N}$ into an actual infinity, so the argument goes, we form an object of ω items – it can be neither more nor less. Then, when we add the punctured neighbourhood of 1, $\lambda = (0,1] - \mathbb{Q}$, we obtain a skeleton of $\omega + 1$ items.

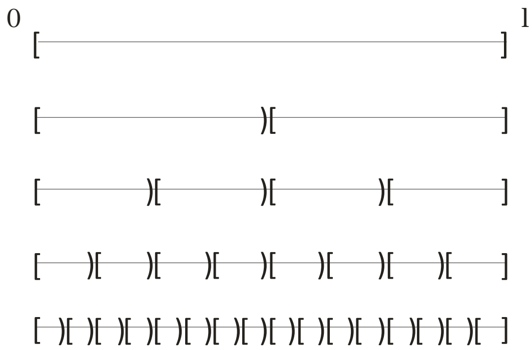
(2) But to show that this is not correct, let us re-examine our two trees – the one of points and the one of intervals.





However, we can factor this as well: $\mathbf{2}^\omega \oplus \mathbf{2}^\omega = \mathbf{2}^\omega(\mathbf{1} \oplus \mathbf{1}) = \mathbf{2}^\omega \otimes \mathbf{2}$. This is equivalent to recombining the two trees so that each point is attached an interval.





This diagram shows that for every point there is an interval. Now this is a complete partition of the unit interval. However, it is a partition that is in the process of infinite generation by a potential infinity. If we complete the partition by cardinal exponentiation we generate not only the skeleton but also the entire Cantor set, complete with all its cardinality $\mathfrak{c} = 2^{\aleph_0}$ points. So we can only use ordinal exponentiation, and the maximum number of points generated across the base is $< \omega$ with cardinality $\aleph_0$ and on completion the width of the base of atoms of the continuum is ω and not . $\omega + 1$.

A weird type of fiction

Strictly speaking the actual infinity is *a useful piece of nonsense, a weird type of fiction*, and "visualising" it is difficult. I am not for one minute arguing that we should leave the "paradise" that

Cantor is said to have created for us, for the actual infinity is plainly necessary to mathematical science, the whole of classical analysis fails without it, and weird though the object is, I see no reason to suppose it is "inconsistent". It works, and works very well. However, the simple caveat is that – all talk of completing an infinity is a kind of fiction, and every now and then that must hit us in the face. The thing that does make intuitive sense, as Aristotle who introduced the distinction did know, is the potential infinite. But that is not enough for analysis: to get anywhere with the continuum we must suppose that our potential infinite can *in some sense or other* can be treated as an actual, complete aggregate, and then we have the mental labour of "getting our heads around it".

The complete Boolean algebra representation of the continuum

The aim here is to combine our two trees model of the continuum into a single representation that uses only the language of first-order set theory. So we treat the skeleton of the continuum to be the structure $\mathbb{N}_\lambda = \mathbb{N} \cup \lambda$ and following the argument of the preceding paragraph, this has ω pieces. Everything we are doing is concerned with building a model of the continuum, and the argument is certainly impossible within the confines of first-order logic and set-theory. We use natural language and terms like "model" and allow our symbols to have *meanings* extraneous to any formal manipulations those symbols might allow. Thus $\mathbb{N}_\lambda = \mathbb{N} \cup \lambda$ is a model of the skeleton of the continuum. Let us review the situation:

$\{1\},\{2\},\{3\},\ldots$	Singleton sets denoting rational points of the unit interval. Their aggregate is $\mathbb{N}_{\mathbb{A}} = \{\{1\}, \{2\}, \{3\}, \ldots\}$.
$\mathbb{N}$	The potentially infinite set of all natural numbers: $\mathbb{N} = \{1\} \cup \{2\} \cup \{3\} \cup \ldots = \{1,2,3,\ldots\}$
λ	A symbol denoting the half open interval $(0,1]$ punctuated by the rational dense subset $\mathbb{Q}$ that may be identified with points generated by the potentially infinite binary tree, $\mathbb{T}$. $\lambda = (0,1] - \mathbb{Q}$.
$\mathbb{N}_\lambda = \mathbb{N} \cup \lambda$	A completed infinity of ω parts representing the skeleton of the unit interval.
Fin	The union of all finite subsets of $\mathbb{N}$ given by: $\{\{1\},\{2\},\{3\},\ldots\{1,2\},\{1,3\},\ldots\{2,3\},\ldots,\{1,2,3\},\ldots\}$ **No infinite set is in the collection.** We identify members of this set with the algebraic numbers and their corresponding points of the continuum.
ω	A completed infinity of ordinals: $\{1,2,3,\ldots\}$
$\mathbf{2}^\omega$	The Cantor set; a representative set for the set of all real numbers, though not identifiable with the continuum itself.

To this collection of concepts we now introduce the co-infinite sets.

Co-infinite sets

The set $\omega = \{\{1\},\{2\},\{3\},\ldots\}$ has no last member, so subtraction from ω is meaningless. However, we can remove sets, finite or infinite, from ω, by the difference of sets. Thus, for example, the co-infinite set $\omega-\{1\}$ is the infinite set that contains every ordinal number except the number 1, and so on. Labelling the atoms of the Cantor set by $\omega_{\mathbb{A}}=\{\{1\},\{2\},\{3\},...\}\cong\omega=\{1,2,3,...\}$ then the set $\{\omega-\{1\},\omega-\{2\},\omega-\{3\},...\}$ is a set of co-atoms of the Cantor set. Now as our model of the skeleton of the continuum is made of ω atoms, with labels: $\{1\},\{2\},\{3\},...,\lambda$, then our co-skeleton shall be made of ω co-atoms with labels $\omega-\{1\},\omega-\{2\},\omega-\{3\},...\ \mu$. The meaning of μ shall be explained below. We shall interpret the other co-atoms as follows: the co-atom $\omega-\{1\}$ will stand for the difference of the open interval $[0,1]$ and the first atom labelled by $\{1\}$ which happens (always in this construction, however the dense subset is constructed) to be 0. That is $\omega-\{1\}$ represents $(0,1]$. It is the unit interval less one point, which in this case is 0. But the removal of 0 from the unit interval is arbitrary – we could have removed any other point and reached an equally valid representation of the unit interval punctuated by a single point. Thus $\omega-\{2\}$ is the interval $[0,1]$ less the second point in the dense subset labelled by $\mathbb{N}_{\mathbb{A}}=\{\{1\},\{2\},\{3\},...\}$. And so on for all the co-atoms. We label the potentially infinite collection of co-atoms by $\mathbb{N}_{\mathbb{B}}=\{\omega-\{1\},\omega-\{2\},\omega-\{3\},...\}$; each one of these represents the unit interval less just one point. Another version of the skeleton is $\mathbb{N}_{\mu}=\mathbb{N}_{\mathbb{B}}\cup\mu$.

The supremum of the set of all finite subsets

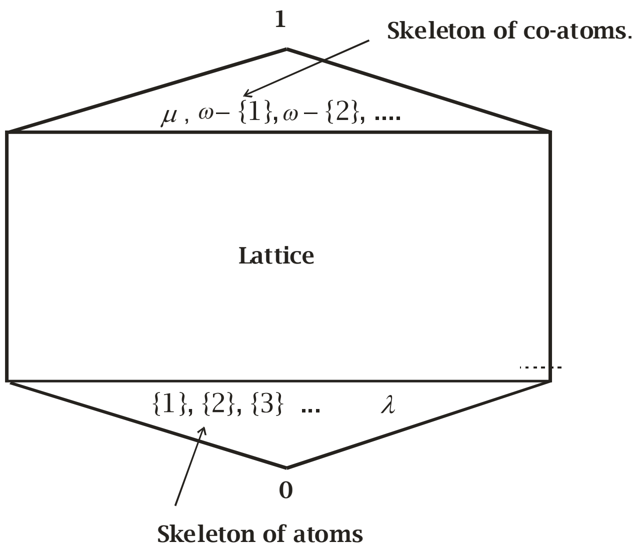
Regarding the symbol μ – it is given by $\mu=\omega-\lambda$ – by analogy with the other operation of removing points. So μ is the supremum of the union of all finite subsets of ω (or $\mathbb{N}$, for they are the same). Now strictly speaking the union of all finite subsets of $\mathbb{N}$ does not have a supremum any more than $\mathbb{N}$ does. However, in set theory we actually define ω to be the supremum of all the ordinals, and by a similar move we may define μ to be the supremum of the set of all finite subsets of $\mathbb{N}$. To form this set from the bottom up we take unions of singleton sets, for example, $\{1,2\}=\{1\}\cup\{2\}$ and proceed iteratively accordingly. Going from the top down (from the co-atoms in the direction down the lattice to the atoms), we take an object and start taking intersections. So μ is a set whose subsets are always finite subsets of $\mathbb{N}$. In one sense, that makes μ identical to $\mathbb{N}\cong<\omega$. It is as if we were generating $\mathbb{N}$ from a single aggregate by taking intersections; but this aggregate cannot be identified with ω because intersections of ω would contain infinite subsets; so $\mu\neq\omega$.

Navigating up and down the lattice

To clarify this point about going down the set, which proceeds by taking intersections, just as going up the set proceeds by taking unions. So taking two co-atoms, say, $\omega-\{1\}$, $\omega-\{2\}$ their intersection is $\omega-\{1\}\cap\omega-\{2\}=\omega-\{1,2\}$. Taking all sum of all the intersections, we obtain: $\lambda=\omega-\{1,2,3,...\}=\omega-\mathbb{N}$, which is what we expect. In our interpretation, this makes λ into a symbol representing the unit interval less the dense subset paired off with the labels $\{1\},\{2\},\{3\},...$. We may also write $\lambda=\omega-\mu$ and $\mu=\omega-\lambda$.

Boolean representing model of the continuum

So now we have a single Boolean algebra with 2^{ω} points representing (but not identical to) the continuum, with atoms: $\{1\},\{2\},\{3\},\ldots\ \lambda$ and co-atoms: $\omega-,\{1\},\ \omega-\{2\},\ \ldots,\mu$. At this stage we can represent the complete lattice of the continuum as:



We have seen that **Fin** is the union of all finite subsets of $\mathbb{N}$:

Fin $= \{\{1\},\{2\},\{3\},\ldots\{1,2\},\{1,3\},\ \ldots\{2,3\},\ldots,\{1,2,3\},\ldots\}$.

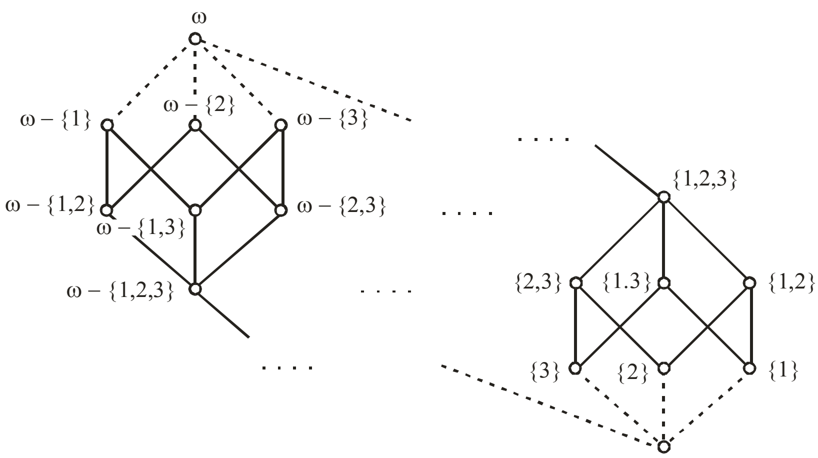
This is isomorphic to $\mathbf{2}^{<\omega}$. So we define **Cofin** to be the intersection of all finite subsets of the co-atoms.

Cofin $= \{\omega-\{1\},\omega-\{2\},\omega-\{3\},\ldots\omega-\{1,2\},\omega-\{1,3\},\ \ldots\omega-\{2,3\},\ldots,\omega-\{1,2,3\},\ldots\}$.

Also **Cofin** is also isomorphic to $\mathbf{2}^{<\omega}$.

The inexhaustible boundary of the Cantor set

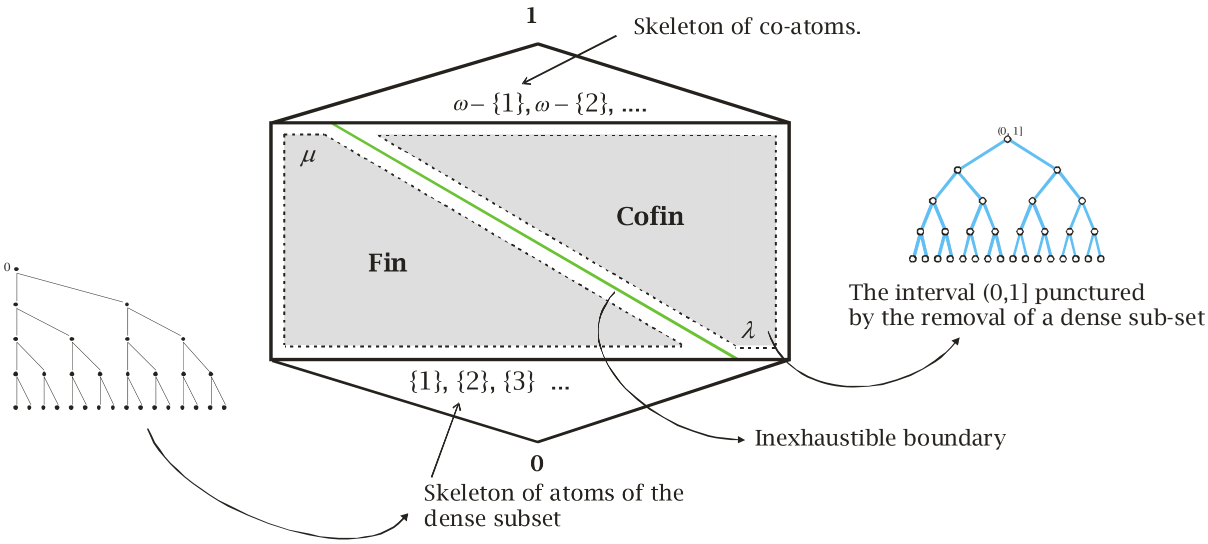
Within the Cantor set $\mathbf{2}^{\omega}$ there is an **inexhaustible boundary**.



By ascending from below from unions of finite sets **we can never reach a set of infinite size**. Conversely, by descending from above by intersections of co-finite sets **we can never reach a set of finite size**. In fact, the boundary is **inexhaustible upon inexhaustible**. We can subtract an infinite set from an infinite set to obtain another infinite set – for example, we can subtract the set of all even numbers from the set of all numbers to obtain the set of all odd numbers, and this process of subtraction can go on and on *ad infinitum*. An infinite set is potentially inexhaustible. No matter how many finite or potentially infinite iterations we make we can never make an infinite set out of a finite one, and conversely never make a finite set out of an infinite one. For this reason, we introduce the notion of a limit, conceptually equivalent to taking an actually infinite process. Then we say that an actually infinite number of intersections produces a point, and an actually infinite union of points produces an interval. This is the very fiction that lies at the heart of analysis as we know it.

Representing the inexhaustible boundary in the Boolean model of the continuum

We modify our diagram of the Boolean lattice of the continuum by introducing a representation of the inexhaustible boundary.



As already indicated $\mathbf{Fin} \cong \mathbf{2}^{<\omega}$ and $\mathbf{Cofin} \cong \mathbf{2}^{<\omega}$, both of size $\aleph_0$ so the diagram is not relative to size. The entire set represents the Cantor set $\mathbf{2}^{\omega}$ of size $2^{\aleph_0}$ so the vast bulk of the numbers corresponding to points of the continuum lie in the boundary, which is here shown only by a line. We have identified the numbers or points of **Fin** with the algebraic numbers; **Cofin** contains no numbers at all, these being sets representing intervals of positive definite measure. *So all the transcendental numbers lie in the boundary, and the boundary is composed of nothing but transcendental numbers.*

The derived set

I call this structure the **derived set**. Taking the skeleton of the continuum to be a model represented by $\mathbb{N}_\lambda = \mathbb{N} \cup \lambda$ the model of the continuum is the exponent of this set:

$$\text{arithmetical continuum} = \mathbf{2}^{\mathbb{N}_\lambda}.$$

I shall use subscript $\mathbb{A}$ to denote the set of atoms of the dense subset of the continuum, and the subscript $\mathbb{B}$ to denote the subset of co-atoms of the unit interval. Then the skeleton is $\mathbb{N}_{\mathbb{A}} \cup \lambda$ and the co-skeleton is $\mathbb{N}_{\mathbb{B}} \cup \mu$. The arithmetical continuum is either $\mathbf{2}^{\mathbb{N}_\lambda} = \mathbf{2}^{\mathbb{N}_{\mathbb{A}} \cup \lambda}$ or $\mathbf{2}^{\mathbb{N}_\mu} = \mathbf{2}^{\mathbb{N}_{\mathbb{B}} \cup \mu}$.

Proof that the derived set is the arithmetical continuum

What is there to prove, and what kind of proof do we expect?

So what at this stage is there to prove? The claim is that the arithmetical continuum is the derived set by cardinal exponentiation of its skeleton, which is a subdivision of the unit interval such that: (1) It contains a part isomorphic to the dense countable set of rational numbers, and also that this set is a potential infinity; we represent this part by $\mathbb{N}_{\mathbb{A}}$; (2) It contains a single part that completes the first part by creating a complete cover for the unit interval; we represent this part by λ; (3) That together the two parts, represented by $\mathbb{N}_{\mathbb{A}} \cup \lambda$, comprise a partition of the unit interval into actually infinite ω pieces; (4) That every other point of the continuum (in the unit interval) is generated by cardinal exponentiation from this skeleton.

What kind of proof?

We make the usual caveat about this proof. In essence, what we have to do is to deduce all of this from the Axiom of Completeness – and we may incorporate any regular standard result of algebra or set theory – that could include the Dirichlet Pigeon Hole principle as well. However, from the outset I remind the reader that the Axiom of Completeness is second-order; that the Dirichlet Pigeon Hole principle is non-formalizable; that the notion that "all mathematics is set-theory" is bogus; and the notion that proofs in analysis can and have been formalized according to some clear and distinct notion, particularly of first-order logic, is also bogus. So I will not be able to pander to illusions, even if I were willing to do so – it's just impossible. In addition to all of this, the statement above refers to distinctions such as that between the potential and actual infinity that have some representation in first-order set theory in the symbols $<\omega$ and ω respectively, but are strictly interpretations that lie in the domain of concepts and meanings rather than mechanical rules or axioms.

Almost nothing left to prove

There is almost nothing left to prove. The relation between the skeleton and its derived set by exponentiation is absolutely standard first-order set theory, or equivalently first-order lattice theory, and may be found in any text-book on the subject. It is a fact that it has been known for at least half a century that the continuum is a lattice. Many of these ideas are enshrined in the first-order treatment of the continuum. So far as first-order theory goes, almost everything that could be said about the continuum has already been said about it. I refer in particular to Bartoszynski, Tomek and Judah, Haim [1995]: *Set Theory on the Structure of the Real Line*, which is a very heavy book indeed (in all senses of the meaning of "heavy") but I believe to be an inspired work that says everything and more about the structure of the real line from within the language of set-theory. The conclusion of this book is that every model whatsoever of the real line, in terms of its internal structure and the relation of the cardinalities, is theoretically possible. Such a conclusion simply means that if we wish to pick one of these models, then we must **go outside set-theory** in order to do so. That invariably means that the choice rests on other principles, and an empirical principle is in the foreground, for the arithmetical continuum is **not an object of phenomenology** – there are aspects of it that do belong to phenomenology, but the concept as a whole belongs, and always has belonged, to the territory of model building. So the question is ultimately empirical.

The role of the Axiom of Completeness

The Axiom of Completeness represents one attempt to solve the empirical question of the structure of the continuum. It has a historically favoured position, and below I shall also argue that it has the advantage in being the simplest model of the continuum that does express a notion of continuity. The discrete, atomistic model of Hume may in some sense be "simpler", but philosophically it is no less challenging and it will be extremely difficult in a technical sense to articulate it.

Proof step 1: the Heine-Borel Theorem

The claim is that the skeleton described above is the immediate consequence of the Completeness Axiom in the form of the Heine-Borel theorem. So first to state that theorem.

> **Heine-Borel theorem of real analysis**
>
> Let *X* be a closed, bounded set on the real line $\mathbb{R}$. Then every collection of open subsets of $\mathbb{R}$ whose union contains *X* has a finite subclass whose union also contains *X*.

By analysing the proof of this theorem we will generate the skeleton of the closed interval as the structure denoted by $\mathbb{N}_{\mathbb{A}} \cup \lambda$ above.

(1) A preliminary issue. A minor point: the theorem does not explicitly state that the closed, bounded set *X* should be a connected set; however, if it is disconnected then it can either be enclosed in a single connected set (as it is bounded) or broken up into a finite collection of closed, bounded sets. In either case, we have only to prove the theorem for a single closed interval, and as all such intervals are isomorphic to $[0,1]$, it is customary to deal with this alone.

(2) About the Heine-Borel theorem. The theorem is concerned with the notion of a cover of an interval.

> **Definition, cover:** A collection of open intervals in $\mathbb{R}$ whose union contains the interval $[a,b]$ is called an *open cover* of $[a,b]$. Given a cover *C* of $[a,b]$, a *subcover* is any collection of sets from *C* whose union also contains $[a,b]$.

The theorem states that any infinite cover of a closed interval has a finite subcover. In order to prove the theorem and conduct the analysis of that proof we should observe the very close parallel between the theorem and the Zeno paradox of the dichotomy. Also the theorem fails if the interval is open, so the distinction between an open and a closed set is crucial to it. **The Dichotomy of Zeno** demonstrates that the half open interval $[0,1)$ does have an infinite cover that cannot have a finite subcover. The infinite cover is the union of the sets:

$$[0,\tfrac{1}{2}),\ [0,\tfrac{3}{4}),\ [0,\tfrac{7}{8}),[0,\tfrac{15}{16}),\ \ldots$$

In this case the collection is a potentially infinite collection, not an actually infinite one. Only by taking the whole actually infinite collection can we cover $[0,1)$. Were we to take the actually infinite collection, as the Axiom of Completeness would allow us to do, then we would add the limit of the sequence $\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{7}{8}, \tfrac{15}{16}, \ldots$, which is 1, and this would then also cover the closed interval $[0,1]$. The Heine-Borel theorem simply looks at this process in reverse.

> **Proof of the Heine-Borel theorem**
>
> Let $[0,1]$ be a closed interval in $\mathbb{R}$. Assume it has a cover Σ – possibly infinite. Then there is a neighbourhood $B = \{x \in [0,1] : 1 - x < \varepsilon, \text{for } \varepsilon \in \mathbb{R}\}$ which is a member of Σ. Removing this

neighbourhood from $[0,1]$ we obtain the interval $[0,1)=[0,1-\varepsilon)$. By defining suprema on finite subsets of this interval, it follows from the Completeness Axiom that there is a finite cover, a subsequence $\Sigma_n, n\in\mathbb{N}$ that covers $[0,1-\varepsilon)$. Then $\Sigma_n \cup B$ is a finite cover for $[0,1]$.

To further explicate this proof: suppose we have a possibly infinite cover Σ of $[0,1]$. Then there is some open interval that is a member of this cover that covers 1. This will be an open interval of the form: $B=(\varepsilon,1]$. We then construct a finite sequence Σ_n of unions from members of the cover such that (a) the first member covers 0; (b) Any two successive members meet at a common point or overlap. Since Σ is a collection of open sets, each member must have a positive size. If we add these sizes together, we see that the collection "grows" towards the point 1. Since every point in $[0,1]$ is covered by some member of the collection, there will be a point in $B=(\varepsilon,1]$ that is overlapped by some member of the collection; another way of putting this is that the supremum of the collection is an element of $B=(\varepsilon,1]$. (To say it were not would contradict the assumption that Σ is a cover.) Denote this supremum by $\sup C$; then $\sup C\in(\varepsilon,1]$. While $\sup C$ is the limit of an infinite sequence, there will be a number $d<\sup C$ such that $d\in(\varepsilon,1]$. While we have to go through an infinite sequence to reach $\sup C$ for some finite $n\in\mathbb{N}$, $d\in\Sigma_n$. So $\Sigma_n\cup B$ is a finite cover for $[0,1]$. In fact, $\sup C=1$ but it is not necessary to assume this for the proof to go through.

Proof step 2: Obtaining the skeleton

From the Heine-Borel theorem we shall obtain the skeleton of the continuum is obtained. The argument is an excursus on the Alexandroff 1-point compactification.

Definition, Alexandroff 1-point compactification

The addition of a neighbourhood of *b* to the open interval $[a,b)$ shall be called the *Alexandroff 1-point compactification* of $[a,b)$.

The Heine-Borel theorem asserts that every closed interval $[a,b]$ in the arithmetical continuum has a one-point compactification. To further complete the argument, we shall also need the Dirichlet Pigeon Hole principle. To get a flavour for how the skeleton emerges from this argument, let us go back to the Dichotomy of Zeno and construct a cover for the half-open interval $[0,1)$ from it. This shall be the collection of half-open intervals: $[0,\tfrac{1}{2})\cup[\tfrac{1}{2},\tfrac{1}{4})\cup[\tfrac{1}{4},\tfrac{1}{8})\cup\ldots$. This collection is a potentially infinite $<\omega$ aggregate that covers $[0,1)$. There is no over-lap between members of the collection, so they are mutually exclusive. They would be "atoms" of the Boolean algebra $2^{<\omega}$ derived by exponentiation from it, saving the fact that $2^{<\omega}$ cannot have atoms because it is non-atomic. But the reason for the non-atomicity is that the collection is potentially infinite. The other pre-condition of atomicity is met – namely, that the atoms do not overlap – they are mutually exclusive. So we can pair off these intervals with members of the set $\mathbb{N}_{\mathbb{A}}=\{\{1\},\{2\},\{3\},\ldots\}$.

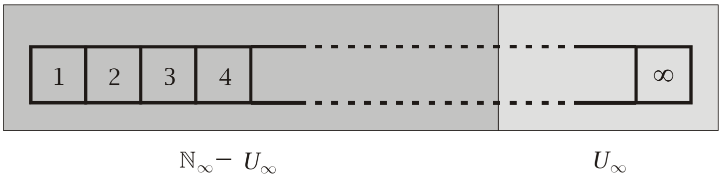
interval	$\left[0,\dfrac{1}{2}\right)$	$\left[\dfrac{1}{2},\dfrac{3}{4}\right)$	...	$\left[\dfrac{2^{n-2}}{2^{n-1}},\dfrac{2^{n-1}}{2^{n}}\right)$	...
label (name)	$\{1\}$	$\{2\}$	...	$\{n\}$	...

So $\mathbb{N}_{\mathbb{A}} = \{\{1\}, \{2\}, \{3\}, \ldots\} \cong \mathbb{N}$ is a discrete space, and consequently has a compactification. The standard compactification of $\mathbb{N}$ may be found in Davey and Priestley [1990].[35]

One-point compactification of a countably discrete space

Let $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$. Let $U \subseteq \mathbb{N}_\infty$. Let T be the topology on $\mathbb{N}_\infty$ given by

$$U \in T \text{ if } \begin{cases} \infty \notin U \\ \infty \in U \text{ and } \mathbb{N}_\infty - U \text{ is finite} \end{cases}$$

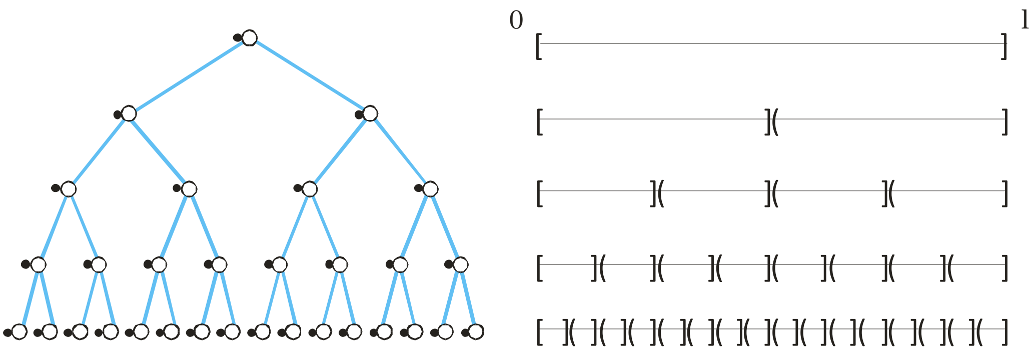


The topology is defined in such a way that, if an open set, U_∞, covers ∞ then what remains, $\mathbb{N} - U_\infty$, must be finite. This mirrors the construction of the finite cover in the Heine-Borel theorem. Clearly, this topology is precisely the same as the skeleton we have proposed for the closed interval $[0,1]$ with the symbol ∞ standing in for what I have denoted by λ for the neighbourhood of 1.

Proof step 3: The one-point compactification

So what remains to be done is to show that this topology is contained within every infinite cover of the closed interval $[0,1]$. To do this, let us construct a canonical atomic cover for $[0,1]$ as follows. Rather than use the rational numbers, we use the countable dense subset $\mathbb{D}$ of $[0,1]$ generated by the binomial tree $\mathbb{T}$ whose nth level is the set $\frac{1}{2^n}, \ldots, \frac{2^{n-1}}{2^n}$. We use this to create a succession of finite partitions of $[0,1]$ into n pieces:

$$\begin{aligned}
&\left[0,\frac{1}{2}\right),\left[\frac{1}{2},1\right]\\
&\left[0,\frac{1}{4}\right),\left[\frac{1}{4},\frac{2}{4}\right),\left[\frac{2}{4},\frac{3}{4}\right),\left[\frac{3}{4},1\right]\\
&\ldots\\
&\left[0,\frac{1}{2^n}\right),\left[\frac{1}{2^n},\frac{2}{2^n}\right),\left[\frac{2}{2^n},\frac{3}{2^n}\right),\ldots,\left[\frac{2^{n-2}}{2^n},\frac{2^{n-1}}{2^n}\right)\left[\frac{2^{n-1}}{2^n},1\right]\\
&\ldots
\end{aligned}$$



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[35] p.197

At each finite stage $n \in \mathbb{N}$ this creates a finite, atomic partition of $[0,1]$ that acts as the skeleton of a Boolean algebra. In the limit as $n \to \infty$ this partition becomes an infinite cover for $[0,1]$. The question is – how many pieces does this skeleton contain? If we think of this tree as the Cantor set, then on completion it has 2^{ω} members, each of which is a real number. However, there are only countably infinite members in a dense set – the collection $\frac{1}{2^n}, \dots, \frac{2^{n-1}}{2^n}$ as $n \to \infty$ is countably infinite. The tree in question is the potential infinite tree $\mathbb{T} \cong \mathbf{2}^{<\omega}$; we envisage that it is this tree that is completed into an actually infinite collection of atoms, just in the manner in which we allow $\omega = \{1,2,3,\dots\}$ to be the actually infinite aggregate of all finite ordinals.[36] So there are ω pieces in this infinite atomic partition; its order type is ω. We now pair off the members of this cover with the members of the set $\mathbb{N} = \{1,2,3,\dots\}$.

(1) A lacuna in the argument, and its resolution. However, there is a lacuna in the argument here, and one may object: since a dense subset *X* of the reals has no first or last element, it is not possible to make the pairing: first element of $X \to \{1\}$. There are several replies to this objections. Firstly, since our theory embraces both set theory and analysis, we are entitled to make use of all the tools available in set theory, of which the Axiom of Choice is one. Subject to that axiom the dense subset $\mathbb{D}$ can be well-ordered, and then we can pair off its elements of under a well-ordering with those of $\mathbb{N} = \{1,2,3,\dots\}$. However, this may seem arbitrary, so we advance a second argument.

(2) Another resolution. This is in a sense a deeper answer. We go back to the point that any actually infinite aggregate is a kind of ideal object or fiction that is problematic, and sometimes those problems "hit one in the face". Now the set $\frac{1}{2^n}, \dots, \frac{2^{n-1}}{2^n}$ for any finite *n* can be well-ordered. The skeleton can only be constructed by assuming that we can take such a sequence to an actually infinite collection by completing the collection $n \to \infty$. The whole motivation for this move is so as to be able to introduce limits into the conceptual apparatus. When we partition the line for finite *n*, the result

$$\left[0,\frac{1}{2^n}\right),\left[\frac{1}{2^n},\frac{2}{2^n}\right),\left[\frac{2}{2^n},\frac{3}{2^n}\right), \dots ,\left[\frac{2^{n-2}}{2^n},\frac{2^{n-1}}{2^n}\right)\left[\frac{2^{n-1}}{2^n},1\right]$$

clearly has a first element. Its function is to model the notion of a particle moving from 0 to 1 so it represents the first part of space that a particle moves into after leaving 0. Now in $\mathbb{Q} \cap [0,1]$ or any other dense subset there is no first element, but in space, as we progress from 0 to 1, there is a first portion of the space that we enter into. The actually infinite dense subset only arises because of our desire to model space, so hence, there must also be a first element to this subset, the next point after 0 that is both in the set and is the next one to be encountered. Hence, subject to the idealised notion that there is a completed infinity at all, the dense subset *X* must have a well-ordering.

(3) Different descriptions. Furthermore, we have already pointed out that the dense subset $\mathbb{Q} \cap [0,1]$ just is the same set as the well-ordered binomial tree $\mathbb{T}$ under a different description. This identity of descriptions cannot be formalised in first-order set theory.

[36] Another way of dealing with this point is that we could say that we generate the entire Boolean lattice (tree) with $\mathbf{2}^{\omega}$ points, and then take the logarithm of this structure to obtain the skeleton of ω atoms. You generate the whole lattice and its skeleton at one and the same time.

(4) One-point compactification of the unit interval. When we complete the set of rational numbers $\mathbb{Q} \cap [0,1]$ by treating it not as a potentially infinite aggregate but as an actually infinite one, we obtain the complete set of atoms Σ that is isomorphic to the one point compactification of a countably discrete space with the topology *T* described above; we denote this structure $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ where ∞ is a neighbourhood of 1 in $[0,1]$.

One-point compactification of the unit interval

Let $\mathbb{N} = \{1,2,3,...\}$ denote a set of representatives of the cover Σ for the unit interval.

Let ∞ be a neighbourhood of 1 in $[0,1]$. Let $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$. Let $U \subseteq \mathbb{N}_\infty$. Let *T* be the topology on $\mathbb{N}_\lambda$ given by $U \in T$ if $\begin{cases} \infty \notin U \\ \infty \in U \text{ and } \mathbb{N}_\infty - U \text{ is finite} \end{cases}$

This mirrors the Alexandroff 1-point compactification and demonstrates the existence of a representation of such a compactification with an atomic basis and order type ω.

(5) The order type of any countably infinite atomic cover of the unit interval. Now an infinite cover for the unit interval may have ordinal type $> \omega$. For definiteness, suppose that we have a cover for $[0,1]$ that has $\omega + 1$ parts. Each part is an open interval of the form (a,b) so we can order this cover in the order of lower bound *a*. It is clear that such a cover could not be atomic. Every number $x \in X \cap [0,1]$ where *X* is a dense subset must belong to one member of the cover. Since there are ω such numbers in $X \cap [0,1]$ when it is looked upon as a completed totality, by the Dirichlet Pigeon-Hole Principle there must be one such number, x_i belonging to two intervals in the cover, and their intersection cannot be empty. Hence these intervals are not mutually exclusive, and the cover cannot be atomic. We have proven the following theorem:

Theorem. Any countably infinite atomic cover for the unit interval must have order type ω.

This proves the uniqueness up to isomorphism of a canonical atomic cover. Hence, we have also shown:

Theorem. The representation of the skeleton of the unit interval is $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ with the topology of the one-point compactification of a countably discrete space given above.

Proof step 4: From the one-point compactification to the two trees model

$\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ here is a skeleton comprising open intervals. In order to recover the double-tree analysis of continuum into points and intervals we need to show that it is equivalent to our model $\mathbb{N}_\lambda = \mathbb{N}_\mathbb{A} \cup \lambda$ where $\mathbb{N}_\mathbb{A}$ is the collection of representatives of the dense elements $x \in X \cap [0,1]$ in order of generation and corresponds to a set of points. However, it is clear that from each interval in $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ can be removed the single such number, leaving a collection of open intervals of measure $\varepsilon > 0$, ε being irrational, and that the union of all of these may be combined into λ denoting the half open interval $(0,1]$ punctuated by the rational dense subset $\mathbb{Q}$. Note, the derived set by exponentiation of $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ is also a double-tree, but the atoms are intervals, and the co-atoms are also intervals, so it is not a convenient representation of the skeleton of the continuum. It follows that the continuum is the derived set by exponentiation of this skeleton $\mathbb{N}_\lambda = \mathbb{N}_\mathbb{A} \cup \lambda$.

The null-meagre decomposition

Looking ahead

We now wish to check the above model for consistency with the known results about the continuum. The main concerns are:

1. The null-meagre decomposition of the real line;
2. The theory of transcendental numbers.

To explain the null-meagre decomposition, we require further background theory.

Perfect, null and meagre sets

All of the following results follow from the Axiom of Completeness.

1. **Limit points, derived set, perfect set, closure and dense.** Given a set *S*, a point *x* is an **accumulation point** (also known as **limit point**) of *S* if every open interval that contains *x* also contains infinitely many points of *S*. The set of accumulation points of *S* is called its **derived set**. A **perfect set** is a set that is its own derived set. The **closure** of a set *S* is union of *S* with all its limit points. The closure of *S* is denoted $\overline{S}$. A perfect set is equal to its own closure. At set $S \subseteq X$ is said to be **dense** in *X* if is the closure of *S*; that is, if $X = \overline{S}$. It is the Axiom of Completeness that guarantees that the set of rationals is dense in the set of reals: $\mathbb{R} = \overline{\mathbb{Q}}$.
2. **Measure.** The **measure** of a set, *S*, denoted $\mu^*(S)$, is defined in such a way that the measure of an interval is equal to its length: $\mu^*(I) = |I|$; that is, if $I = [a,b]$ or $I = (a,b)$ then $\mu^*(I) = b - a$.[37] We do not here go into the details of the definition of measure; it is sufficient to know that the measure of an interval is equal to its length, which is also called its **outer content**. It is a **result** that the Cantor set is perfect, nowhere dense and has outer content equal to zero.
3. **Null sets.** A set $A \supset \mathbb{R}$ is called a **null set** (or a **measure zero set**) if it is essentially "like" a set with zero measure. That is singletons are null sets and any subset of a null set is a null set. Any countable union of null sets is a null set. (Technically, for each $\varepsilon > 0$ there exists a sequence of intervals I_n such that $A \subset \bigcup I_n$ and $\sum |I_n| < \varepsilon$.)
4. **Nowhere dense.** A nowhere dense set in a topological space is a set whose closure has empty interior.
5. **First category or meagre.** A set is said to be **first category** or **meagre** if it can be represented as a countable union of nowhere dense sets. A subset of *R* that cannot be so represented is said to be of **second category**. **Baire's theorem** states that the complement of any set of first category on the line is dense. No interval in $\mathbb{R}$ is of first category. The intersection of any sequence of dense open sets is dense.

[37] μ is used here to denote the principal element of the prime ideal of all finite sets, $\mathrm{M} = \mathbf{P}(\mu)$, $\mu = \{\lambda\}'$. μ^* is used here to denote the measure of a set.

A gloss on the meaning of null and meagre

The concepts of null, not-null and meagre sets can be confusing – it may be hard to get one's head around them. So some comments: firstly, as regards not-null sets these are sets that have some measure, and are hence extended. So they are like intervals. In order to have an extension a set must have an interior, so open and closed intervals are the paradigm of not-null sets. Thus null sets are sets that have no interiors and are hence things like point sets or boundaries that have lost their interiors. To be dense is to have a closure that completes and equals the parent set. All of these are relations between sets – between a subspace and its parent space rather than properties of the parent set alone. The set of rationals is a set of singleton disconnected points, but its closure is the whole set of reals; so the rationals are not meagre. There are not enough points in any part of a meagre set to create the whole space, to converge upon it in the limit. So meagre sets are things like the boundaries of open sets, but not enough of them, or disconnected in some way to be unable to grow into the space, hence perforated by a dense set. It turns out, as we shall see below, that a meagre set can have non-zero measure, so in some sense be extended in space. This seems like a paradox – it is certainly something that has to be explained. What in effect it means is that the boundaries of open sets, when taken altogether, can have a measurable size, and open sets, when shrunk down to a limit, can reach something of zero measure. So how could that be possible?

* **Independent concepts.** The two concepts of null and meagre are independent. A set can be both meagre and null, like a collection of isolated points. It can be meagre and (not-null) extended, as in the construction of the set in the theorem to follow. It can null and not-meagre, like the dense subset $\mathbb{Q}$. Every countably infinite set must be either meagre or null or both.

The null-meagre decomposition

The symbols $\mathcal{N}$ to denote the **ideal of all null sets** in $\mathbb{R}$. and $\mathcal{M}$ to denote the **ideal of all meagre sets** in $\mathbb{R}$ are standard. Here the term "ideal" indicates the union of all sets of a given category. We met the ideal **fin**, which was the union of all finite subsets of natural numbers. We can proceed to state the main result.

> **Theorem, meagre-null decomposition**
>
> The line can be decomposed into two complementary sets A and B such that A is of first category and B is of measure zero.[38]
>
> **Corollary**
>
> Every subset of the line can be represented as the union of a null set and a set of first category.

---

[38] **Proof of the theorem [Oxtoby pp 4, 5]**: There exist sets $A \in \mathcal{N}$ and $B \in \mathcal{M}$ such that $A \cup B = \mathbb{R}$. Let $\langle q_n : n \in \omega \rangle$ denote an enumeration of rationals according to some rule. Let $I_{i,j} = \left(q_i - \dfrac{1}{2^{i+j}}, q_i + \dfrac{1}{2^{i+j}} \right)$. Let $G_j = \bigcup_{i=1}^{\infty} I_{i,j}$ $(j = 1, 2, \ldots)$ and $B = \bigcap_{j=1}^{\infty} G_j$. For any $\varepsilon > 0$ we can choose j so that $\dfrac{1}{2^j} < \varepsilon$. Then $B \subset \bigcup_i I_{i,j}$ and $\sum_i |I_{i,j}| = \sum_i \dfrac{1}{2^{i+j}} = \dfrac{1}{2^j} < \varepsilon$. Hence B is a null set. However, G_j is a dense open subset of $\mathbb{R}$ being the union of a sequence of open intervals; it also includes all rational points. Hence its complement $\overline{G_j}$ is nowhere dense. Hence, $A = \overline{B} = \bigcup_j \overline{G_j}$ is meagre. That is, $A \in \mathcal{N}$ and $B = \mathbb{R} - A \in \mathcal{M}$.

The essential idea of the proof is to trap every rational number inside an individual open interval which can be shrunk to a size as small as one pleases. These open sets comprise intervals that at any finite stage of the process overlap and cover the unit interval. In the limit they shrink to null sets and uncover a meagre set that has measure 1.[39]

Two kinds of transcendental number

The irrational numbers may be sub-divided into (a) algebraic irrationals, and (b) transcendental reals. Since algebraic numbers correspond to lattice points of **Fin**, this entails that any transcendental number may be classified as: -

1. A transcendental real not belonging to the meagre set *B* in the above decomposition. Hence, belonging to the set A.
2. A transcendental real not belong to the null *A* set in the above decomposition. Hence, belonging to the set *B*.

The first kind of transcendental real shares with the algebraic numbers and the rational numbers the property of having zero measure, so together these points shall be called **zero measure points**. The second kind of transcendental real is an **infinitesimal**. So our model implies that the real line is composed of two kinds of points:

zero measure points	points belonging to the null set *A*
infinitesimals	points belonging to the meagre set *B*

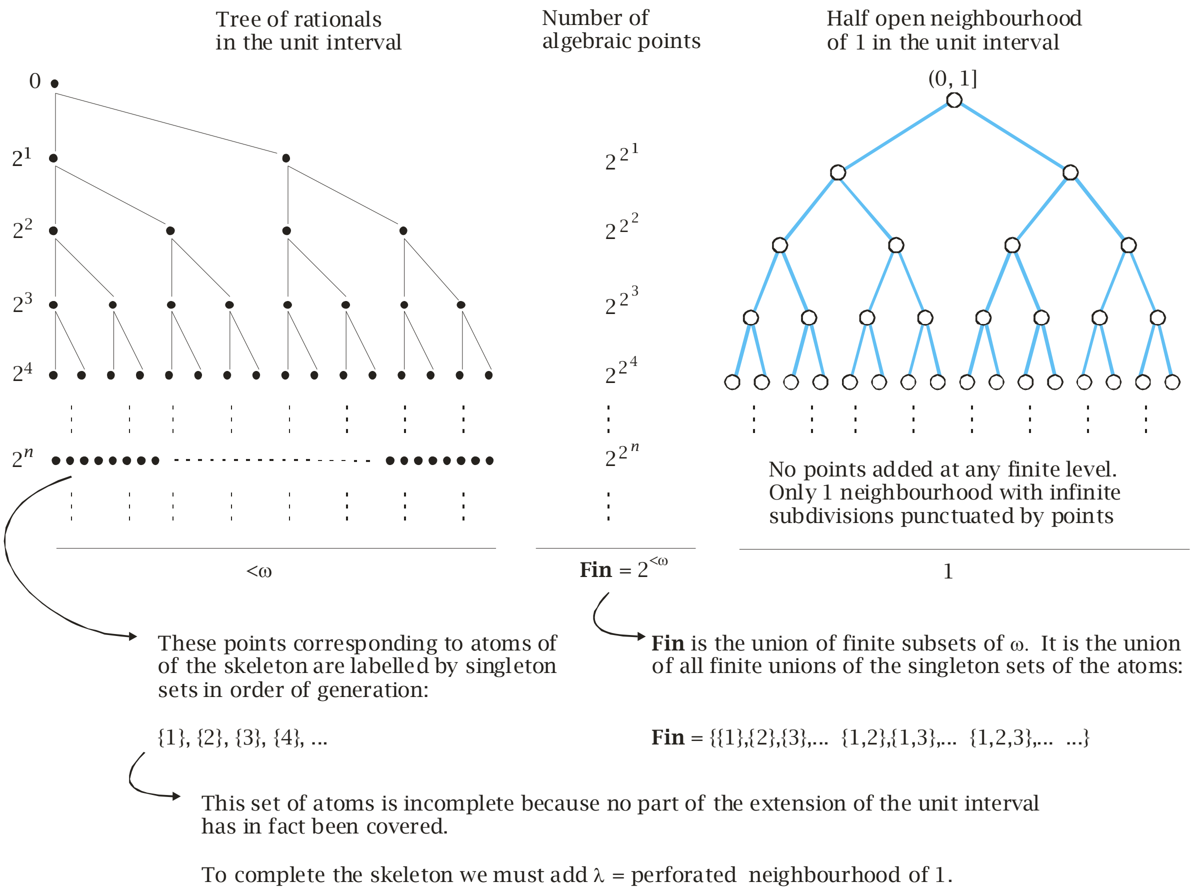
The meagre-null set decomposition in our model of the continuum as the derived set.

We need to account for the meagre-null decomposition in terms of our model of the continuum. Measure zero sets and sets of first category are "small", but in different senses. This situation is readily accounted for by our model.

[39] **Analysis of the proof.** The first thing to remark about the proof is that it is very close to the manner above in which we constructed our atomic cover for the unit interval, and is leads to an equivalent construction for it. To further clarify this: in the proof every rational is enclosed in an open set of size $\frac{1}{2^j}$ where $j \to \infty$. What we are doing is enclosing every rational number within an open interval G_j of a size that can be made arbitrarily small, $\mu^*(G_j) < \varepsilon$. This in the unit interval then gives us a family of a countably infinite dense open subsets of $\mathbb{R}$ and *A* is the union of these such that $\mu^*(A \cap [0,1]) = 0$. Then its complement $B = [0,1] - A$ is a countably infinite set of (closed) nowhere dense subsets of $[0,1]$, and so is meagre; it measure is $\mu^*(B \cap [0,1]) = 1$. The family of open intervals, G_j, has cardinality $\aleph_0$, but both *A* and *B* are unions of members of these families; the cardinality of each G_j is continuum, so the cardinality of both the *A* and *B* sets is continuum. This decomposition demonstrates that a meagre set may have positive measure. Thus we have: -

A	A null set	$\mu^*(A \cap [0,1]) = 0$
B	A meagre set	$\mu^*(B \cap [0,1]) = 1$

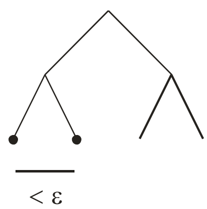
Sets *A* and *B* are themselves collections of $\mathfrak{c} = 2^{\aleph_0}$, continuum many points. Set *A* may be thought of as a collection of subsets, each of which is "centred" one some unique rational number; we shall call these **clusters**. This is what arises from the decomposition. However, the members of these clusters are real (specifically, irrational) numbers that do not belong to the meagre set *B*. The members of *B* are real numbers not belonging to the null set *A*.



On the left hand tree we have the generation of the countably infinite dense subset of the real line, so this set is null and dense; it is the paradigm of the null set in the decomposition. On the right-hand side we are progressively bisecting the unit interval by points of that dense subset. At each finite stage what remains is a finite union of non-null sets – in fact intervals whose measure collectively is 1. But in the limit, once we have punctured this with a set of points taken from the dense sub-set, we get a meagre set, so the set we have labelled λ is the paradigm of a meagre but not null set.

Points of zero measure

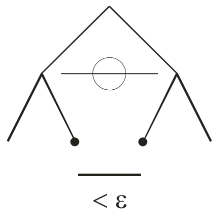
When we take the limit of both trees and make them actually infinite, we generate the transcendental reals. Now take two adjacent branches of the left hand tree of boundary points.



At the *n*th finite stage both of these boundary points corresponds to a dyadic number. The space in between them belongs to the extended part of the unit interval. A transcendental number belongs to the open set $\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right)$ for some $k \in \mathbb{N}$. In the limit as $n \to \infty$ the width of the open interval $\frac{1}{2^n} < \varepsilon$ for all $\varepsilon \in \mathbb{R}$. So the transcendental point between these two branches is a zero measure point.

The infinitesimals

To find the infinitesimals consider the interval between branches of the tree of boundary points that are adjacent in a different sense.

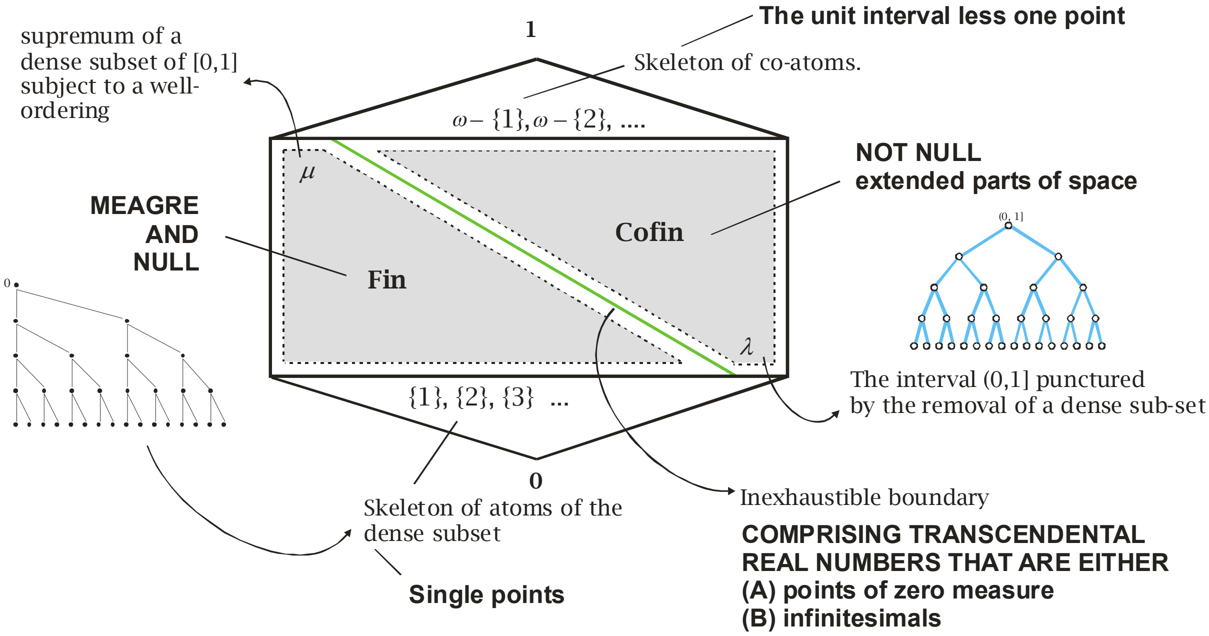


Just as before we shall suppose that the measure between the two branches tends to zero in the limit. However, since both rational points have not been included in the limit of the open intervals in the first case, they must be included now. So this kind of converging branch is the convergence of nested intervals of closed sets of the kind $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$ for some $k \in \mathbb{N}$. Now this set is also squeezed together by the intervals to its left and right: $\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)$ and $\left(\frac{k+1}{2^n}, \frac{k+2}{2^n}\right)$. So this nested sequence has the character of a closed set squeezed between open sets. We may visualise it as: $\left(\frac{k-1}{2^n}, \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right], \frac{k+2}{2^n}\right)$.

So the object $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$ is still an interval with a boundary and an interior not equal to its boundary, and on the principle that extension is never destroyed, even when the tree is actually infinite, the limit still has an extension, though one that is incommensurable with 0. Furthermore, since the limit has its own boundaries, its closure is equal to itself, so no collection of such points could be dense in the unit interval for being already closed it cannot "expand" into whatever is left over of the continuum. This contrasts with the zero measure points. When we take the closure of those, we add back the boundaries, and hence all the points of the line. Since the tree of intervals that generates both kinds of point is potentially infinite, so of cardinality $\aleph_0$, so there is a countable collection of sets containing these infinitesimals that is nowhere dense.

Modified diagram of the Derived Set

We modify our diagram of the Derived Set to incorporate this information.



Transcendental numbers

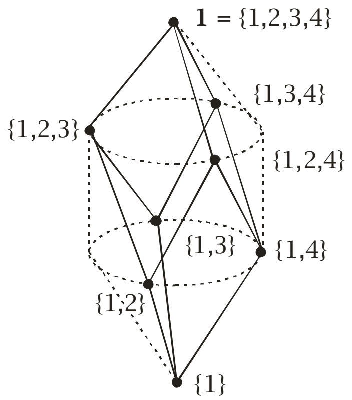
Transcendental numbers and first-order set theory

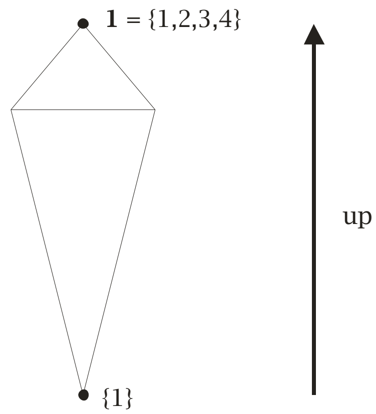
We wish also to be able to relate the theory of transcendental numbers to first-order set theory via the model of the continuum as the Derived Set of $\mathbb{N}_\lambda = \mathbb{N}_\mathbb{A} \cup \lambda$. It is surprising that hitherto no one appears to have considered this issue. First-order set theory was developed to model the continuum, and yet there has been no explicit discussion within it as to what sets the transcendental reals are. However, as already pointed out, set theory does not have any actual real numbers, but it does have real-number generators called ultrafilters. The distinction between a real number and an ultrafilter shall be clarified below. Ultrafilters are generated by devices called forcings.

Filters and ultrafilters

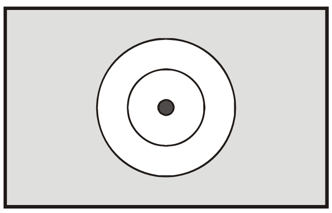
A **filter** is an "up" set. Every filter contains the maximal element, **1**. A filter is **principal** if it has a single lowest point. Not every filter is principal.

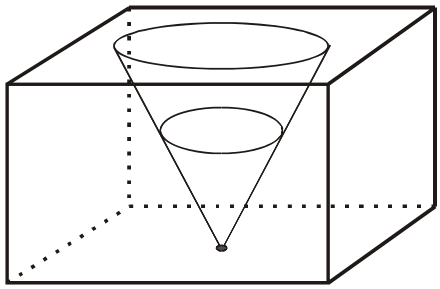
> **Example**
>
> Consider the Boolean lattice $\mathbf{2}^4$ with skeleton: $\{\{1\},\{2\},\{3\},\{4\}\}$. We may define on this lattice a family of those sets that contain as a subset the atom $\{1\}$. This shall be called its **filter**.
>
> filter of atom $\{1\} = \{x \subseteq X : \{1\} \subseteq x\} = \{\{1\},\{1,2\},\{1,3\},\{1,4\},\{1,2,3\},\{1,3,4\},\{1,2,3,4\}\}$.





A filter is also called an ***up-set***, which helps one to visualise its meaning. A filter is said to be a **principal filter** if it has a unique lowest element. In the above example $\{1\}$ is the principal element. This means that every member of the filter contains $\{1\}$ as an element. In finite lattices all filters are principal; however, in infinite lattices this may or may not be the case. The dual notion is an **ideal**, which is a *down-set*. If we visualise the entire lattice as a "space", a principal filter corresponds to a series of concentric circles radiating from a specific point of this space. A filter may also be pictured as a section through the space.





*These diagrams are for heuristic purposes. Every filter contains the entire space, so these diagrams do not show the greatest, **1**, and least, **0**, elements of the lattice.*

In a finite lattice a filter is called an **ultrafilter** if its principal element is an atom. In an infinite lattice there may not be any principal elements; an ultrafilter in an infinite lattice is one which covers the **0** element. So in infinite lattices ultrafilters can take on the role of atoms.

> **Prime ideal**
> The dual notion is that of a **prime ideal**. An ideal is **prime** if it is only covered by the greatest element, **1**, of the lattice.

Real numbers and ultrafilters

In set theory real numbers are generated by ultrafilters. However, the ultrafilters lack principle elements, so the real numbers themselves are not added, only their generators. Although the ultrafilters lack principle elements, they may themselves be shown to form atoms of a Boolean lattice. So the ultrafilters do almost everything that is required of a real number. In analysis real numbers are generated by Cauchy sequences, which are equivalent to nested intervals, and the Axiom of Completeness explicitly adds the real numbers as the limits of those sequences when those sequences are understood to be actually infinite completed totalities. So this highlights the differences between the two theories. To add real numbers as principle elements of ultrafilters from within set-theory, one

would have to quantify over ultrafilters and then make an existential statement, so this would be a second-order statement; hence, first-order set theory simply ignores real numbers and treats of ultrafilters instead.
* **The obvious question.** The obvious question is – given a second order language, what can we say about the relationship between ultrafilters and real numbers?

Ultrafilters, generic sets and forcing arguments
Generic sets constructed by forcing arguments were introduced by Cohen [1966] in order to prove that there is a model of set theory (ZF)[40] in which there are non-constructible sets, and that the Axiom of Choice is independent of the axioms of ZF set theory. Generic sets are special kinds of ultrafilters that are so strange that they could not be constructed out of the definitions and axioms available to ZF set theory. To construct them Paul Cohen introduced the set-theoretic tool of forcing.

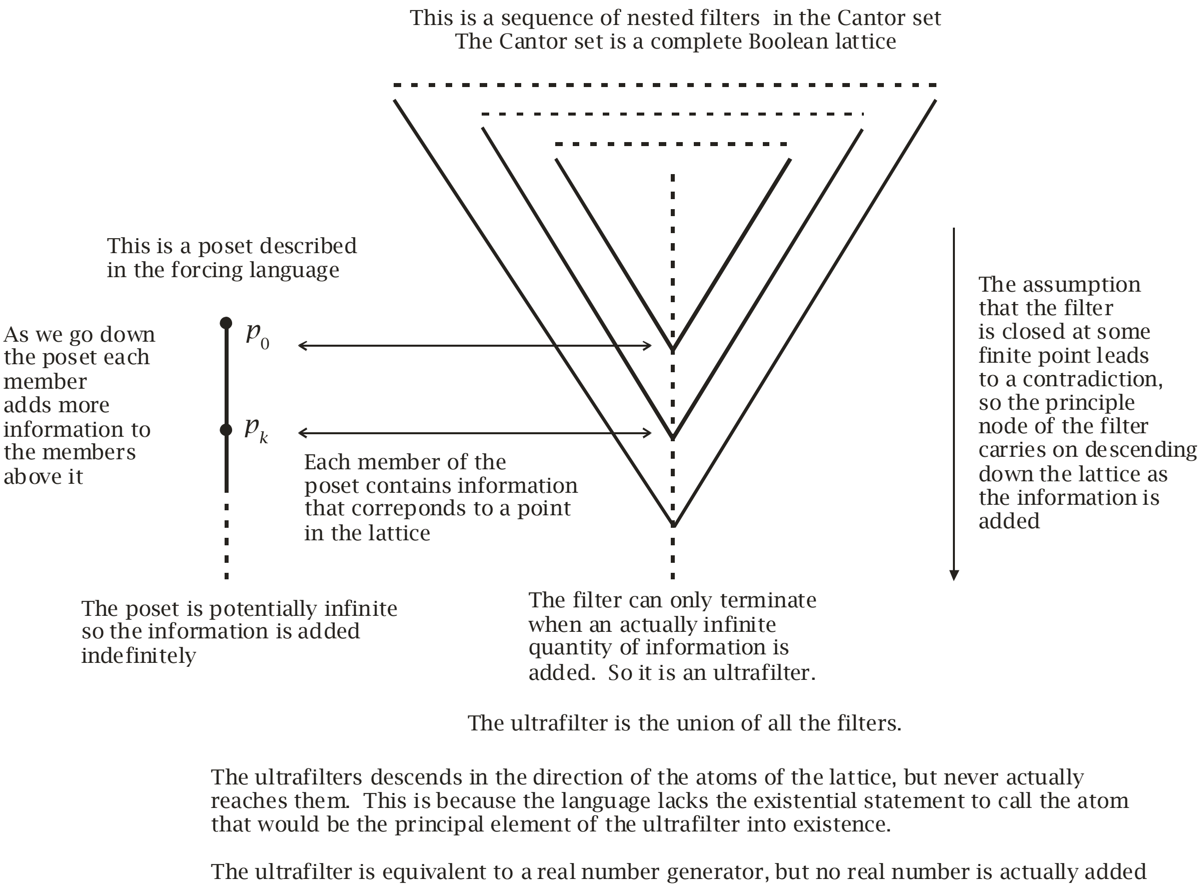
A potted history of forcing arguments
(1) Forcing arguments present many difficulties to understanding, so what one needs is a principle to see the wood from the trees. In essence a forcing argument is not very different from the kind of argument that became available to the Greeks around the time of Plato to demonstrate the existence of irrational numbers. One assumes, for example, that $\sqrt{2}$ is a rational number, and derives from that assumption a contradiction. Something fundamentally very similar is at work in a forcing argument. An ultrafilter is in essence an infinitely descending filter within a Boolean algebra – it is a set of sets that converges towards the atoms of the algebra. But if the algebra is non-atomic, then it cannot reach an atom, it can only get infinitely closer and closer to an atom. Let us define a certain kind of object, call it a "generic set", and ask, could this object be constructed at some finite stage within an ultrafilter? Then this assumption is shown to lead to a contradiction, so the conclusion is that the generic set is an ultrafilter. If that generic set happens to be undefinable in the language of ZF set theory, then this shows that not all objects in ZF are constructible.
(2) **The hypothetical character of forcing arguments.** There is an immediate caveat, for the method did not actually show that the generic sets exist, or form part of set theory, or of the continuum for that matter, the method only shows that *if they exist, then they are not constructible.* So **the method proves absolutely all things to all men**, or rather that ZF is an incomplete theory, and may be supplemented in all sorts of different ways.
(3) **Cohen forcing.** By way of example, consider the simplest kind of forcing argument there is:

> **Definition, Cohen forcing**
> The set of all finite partial functions from ω to $\mathbf{2}$ subject to a partial order from which one could deduce a contradiction from the assumption that the ultrafilter defined by that partial order can terminate at finite point is called Cohen forcing. It is denoted: $\mathbb{C} = (\mathrm{Fn}(\omega,2),\subseteq)$. The symbol $\mathrm{Fn}(\omega,2)$ denotes the set of all finite partial functions from ω to $\mathbf{2} = \{0,1\}$.

To explain: $\mathrm{Fn}(\omega,2),\subseteq$, is a partial order such that at any given stage there are two disjoint finite subsets of ω , $p^{-1}(1)$ and $p^{-1}(0)$, to be interpreted as meaning that $p^{-1}(1)$ is the set of true

[40] ZF = Zermelo-Fraenkel set theory. ZFC = Zermelo-Fraenkel set theory with the Axiom of Choice.

statements about the filter, and $p^{-1}(0)$ is the set of false statements about the filter. As we proceed down the partial order, we add more information, so we have a subset relationship $p_o \subseteq p_1 \subseteq p_2 \subseteq \ldots$.



Suppose $p_0 = (\{3,47\},\{932\})$ this means $3 \to 1,\ 47 \to 1,\ 932 \to 0$; that is 3 and 47 are in the filter, but 932 is not.[41] To construct the filter we create a point in the Boolean lattice that corresponds to the conjunction of $3 \in \text{filter}, 47 \in \text{filter},\ 932 \notin \text{filter}$ and generate the whole filter by the rule that this point is included in it. Since this is a conjunction it lies as a lattice point below each of $3 \in \text{filter}, 47 \in \text{filter},\ 932 \notin \text{filter}$ taken separately.[42] Say $p_1 = (\{3,47,92\},\{932\})$ so $p_0 \subseteq p_1$, we conjoin to the above the statement $92 \in \text{filter}$ and so move the bottom of the filter down the lattice towards the atoms of the lattice. Since the lattice is non-atomic, we can never actually reach its atoms, so we can carry on adding information indefinitely. As the partial order is infinite in length – that is dense, meaning in this case never terminating – the union of all the filters created has to be an ultrafilter. We define a total function whose image set is the ultrafilter – it is the map given by $n \to n$th stage in the generation of the filter.

[41] Here $p_0 = (\{3,47\},\{932\})$ just represents an arbitrary starting point for the construction. It does not express the idea that filter corresponding to the whole poset cannot be constructed after some finite point. For that we need another condition, a forcing condition, that effectively says that any attempt to construct it at a finite point leads to a contradiction.

[42] Conjunctions of statements correspond to joins of lattice point and similarly to intersections of sets. In the lattice diagram the join of two points lies **below** the two points themselves. So filters are going down the Cantor set; that also means that the filter, which is the union of all the lattice points it contains, is getting bigger and bigger.

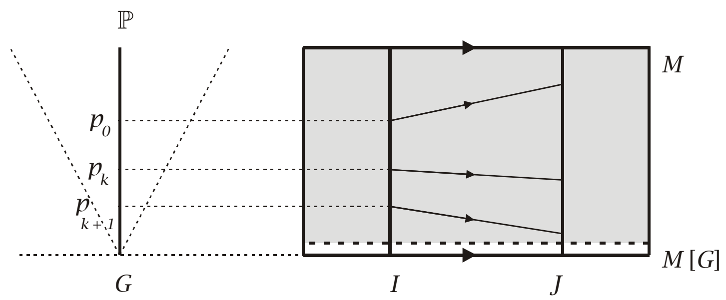
(4) In conclusion, the method of (Cohen) forcing just describes adds to the ground model an infinite subset of ω that was not defined before. It may, hence, be regarded as a real number generator, though the term "real number generator" does not appear in first-order set theory; it is an interpretation from outside the theory. The argument is called a "forcing argument" because it forces the generic object so defined to be an ultrafilter because the assumption that it can be constructed at any finite stage leads to a contradiction.

(5) **The underlying similarity of all forcing arguments.** At some early stage it was realised that all forcing arguments have an underlying similarity. The language used to describe a forcing always constructs a poset (partially ordered set). So it is the properties of the poset that are crucial in determining the structure of the model. We additionally assume that any given poset is "generic" – which here means, that the objects the poset defines in the model are ultrafilters that cannot be completed at any finite level – so they are potentially infinite structures. So the complicated arguments required to demonstrate that the poset is generic can be dropped, and all one has to do is look at the poset and the type of mapping it defines. Additionally, one thinks of this as a three-stage process.

> Stage 1: Let M be a ground model for set-theory.
>
> Stage 2: Let the poset $\mathbb{P}$ of such and such a type define a process for generating a generic set not otherwise included in M. Suppose that this process is completed and that this generic set is added to the ground model of stage 1. Call this the generic extension, denoted $M[G]$.
>
> Stage 3: Deduce the consequences for the generic extension: show that something given in M is false in $M[G]$.

Hence, the general form of a forcing argument is a poset $\mathbb{P} = (\mathrm{Fn}(I,J),\leq)$ where $\mathrm{Fn}(I,J)$ denotes the set of all partial functions from I to J. Using this device we can pretty well prove any relationship between infinite cardinalities.

(6) **Proving all things to all men.** The partial order $\mathbb{P} = (\mathrm{Fn}(I,J),\leq)$ is a potentially infinite chain of conditions that create a mapping from the set I onto the set J.



If the partial order is dense, and hence below any point there lies a (potentially) infinite chain, then if we exhaust the chain (by taking the actual infinite) then we must exhaust the range as well. Initially, J may be a set seen from the ground model M has having larger cardinality than I. The addition of the partial order – equivalent to a real number generator – maps some of I onto all of J, and so establishes a one-one correspondence between them. In the absence of the real number generator, J cannot be compared with I except to say that it is larger, so it has larger cardinality. The addition of the generator gives them the same cardinality in the larger model $M[G]$.

(7) **Cardinality is not absolute.** Thus the notion of a cardinal is not absolute for *M*. Let κ be an uncountable cardinal of *M*. Let $\mathbb{P} = \mathrm{Fn}(\omega,\kappa)$ and let *G* be $\mathbb{P}$-generic over *M*. Then the ultrafilter defined by *G* is a member of the generic extension $M[G]$ and *G* is a function from ω onto κ. Hence κ is a countable ordinal in $M[G]$. In this argument we start with a sequence of functions in a poset, $p_0, \ldots, p_k, \ldots$ that maps part of ω into an ordinal $\kappa > \omega$. We append to *M* a generic function, the equivalent of a real number generator, which the lemma shows maps all of ω onto κ; so we have a bijection in $M[G]$ between these two ordinals; the larger ordinal is said to have been collapsed.

(8) **The fecundity of set theory – it gives birth to all things.** Again, the obvious point is that this is all conditional. We have not demonstrated any particular method of forcing that adds any functions, simply that it is consistent with ZFC to do so. This illustrates the fecundity of ZFC set theory, or alternatively, its weakness. It spawns many models. It is not categorical. A plethora of forcing arguments emerge from which it becomes possible to prove almost any relationship whatsoever between, for instance, two cardinal numbers. For example, one can construct models in which $\aleph_0 < \aleph_1 < \aleph_2 = 2^{\aleph_0}$. Sub-structures that might exist in the continuum can be defined by using one forcing argument or another.

(9) **The search for something more.** Since some of these possible objects lie in direct contradiction with others, and none of them are entailed by the axioms of ZF anyway, this focuses the research on what kinds of additional axioms could be provided. The problem is to find some kind of intuitive reason for adopting one axiom rather than another. Additionally, the problem focuses attention on the fundamentally empirical nature of the continuum, since it is not a structure given to phenomenology or mathematical intuition, it is reasonable to assume that it could have many models, each governed by one set of axioms or another, and that the choice of model will depend on empirical confirmation of results. But so far this program has not been executed, at least not with any universal assent. Meanwhile, more work is being done on multiplying the models, developing more forcing arguments and more potential candidates for axioms.

Generic sets and transcendental reals

The aim is to confirm our structure of the continuum subject to the Completeness Axiom by showing that transcendental numbers are generated by forcing arguments. The Mahler classification of real numbers belongs to the C19th tradition. It divides real numbers into:

1. Algebraic numbers *A*; these include rational numbers, $\mathbb{Q}$
2. Three kinds of transcendental numbers, *S, T* and *U.*

These are all the real numbers that there are.

Outline of the program.

In order to demonstrate that transcendental numbers correspond to ultrafilters generated by a forcing argument, we have to analyse the proof of the Mahler classification – or one should say proofs, because there is a sequence of results that starts with Liouville's Theorem leads among other things to the definition of a Liouville numbers, would work through an analysis of Hermite's proof that *e* is transcendental, proceeds to the Dirichlet theorem providing a condition for the irrationality of a number, then to the extension of the Direchlet theorem analysis of which leads to the Mahler classification. All of this work is heavy, and hence contained in an appendix.

* In this section of the argument if the reader wishes to gain an impression of what is going on without getting bogged down in details, then just look at the diagrams to get started. They really **show** the two different kinds of transcendental. The first as an open set squeezed between two boundaries not included in it and in the limit becoming a point of zero measure. The second as a closed set with its boundaries squeezed by open sets to eitehr side of it, and hence in the limit becoming an infinitesimal. There is consistency and confirmation between the three parts of the theory presented here: (1) The model of the continuum as the Derived Set, (2) The null-meagre decomposition and here (3) the classication of the transcendental numbers.

Points and infinitesimals in the Mahler classification

The transcendental numbers known as U numbers are identified with Liouville numbers in the Mahler classification. There are constructive proofs that the measure of the sets of A, U and T numbers are zero. Hence the measure of the S numbers in the unit interval is 1. We say that **almost all** numbers are in a particular set X if the complement $\mathbb{C} - X$ is a set of measure zero.

Theorem

Almost all numbers are transcendental.

Corollary

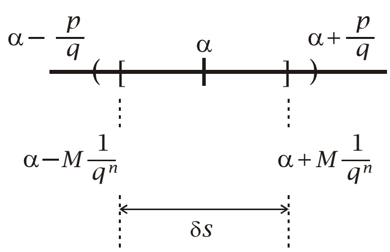
Almost all numbers are S-numbers.

$$m(A)+m(S)+m(T)+m(U)=m([0,1])=1 \quad \Rightarrow \quad m(A)+m(T)+m(U)=0 \quad \Rightarrow \quad m(S)=1$$

It is appropriate to relate this result to our model of the continuum, where we have two kinds of transcendental number: zero measure points, and infinitesimals. This much has already been implied by our arguments with the two trees model. The tree of points when completed into an actual infinite collection adds transcendental reals of zero measure, and hence, does not diminish by one iota the measure of the interval – they are mere boundaries only. So when these and the other points of the tree of points are removed from the unit interval a set of transcendental reals of unit measure must be left behind. It remains to show that analysis of the proofs of the theorem leading up to the Mahler classification would show that the U and T numbers have no measure and are points of zero measure, while the S numbers are infinitesimals. We can do some of that work here and now.

Criterion for algebraic numbers

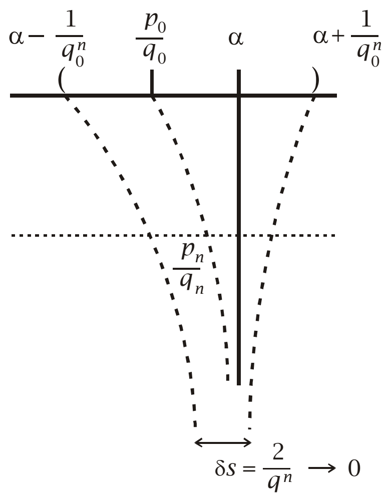
Firstly, a **Criterion for algebraic numbers** is as follows: There exists a $c = c(\alpha)$ such that $\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^n}$ for all rational numbers, $\frac{p}{q} \in \mathbb{Q}$. It is customary to write $M = \frac{1}{c}$ and express this condition as, $\left|\alpha - \frac{p}{q}\right| > \frac{1}{Mq^n}$.



Any algebraic number is "pushed apart" from its nearest rational approximation by an interval.

Criterion for the Liouville numbers

Liouville numbers are the *U* numbers of the Mahler classification. For Liouville numbers instead of being pushed apart from any rational approximation, there is a sequence of rational approximations $\frac{p_n}{q_n}$ to any Liouville number that is squeezed together with it.



Theorem, Every Liouville number is a zero measure point

There will always be two nearest approximations to a Liouville number α one lying below it and one above, both with denominator q_n. These both converge on α without ever becoming identical to it. The result above shows that they are squeezed together by an open set $\left(\alpha - \frac{1}{(q_n)^n}, \alpha + \frac{1}{(q_n)^n}\right)$. In the limit the measure of this set becomes zero – in the limit it is no longer an open set, but a limit point of a nested sequence of open sets. We think of the open set between the two branches as shrunk to a set of zero measure in the limit. The Liouville number lies within any open set in the sequence of open intervals, and is not identical to either of its boundaries. The sequence $\frac{p_n}{q_n}$ in the corollary to Liouville's theorem is a sequence of boundary points, and comprises a meagre set, $\left\{\frac{p_n}{q_n} : n \in \mathbb{N}\right\}$. Hence this sequence defines a transcendental number α not belonging to this (or any other) meagre set. Hence, any Liouville *U* number is a zero measure point. We can show by a similar analysis that any *T* number is also a zero measure point.

The *S* numbers are infinitesimals

We wish to prove that the *S* numbers are infinitesimals. Recall that we showed earlier that an infinitesimal will be obtained by squeezing a closed interval by a limiting process between open intervals, so that it retains a boundary distinct from its interior, and hence is an extension, but one that cannot be measured. To demonstrate this entirely would require much analysis of the argument by which the Mahler classification is constructed, but we can give a strong indication of how this is done. The Mahler classification of transcendental numbers requires:

Theorem, Diophantine condition for irrationality

Let $\tau \in \mathbb{R}$ be irrational. Then there exists a number $\omega(\tau)$ such that τ is irrational iff $\omega(\tau) \neq 0$

For the sake of the argument here it is not necessary at this point to know how $\omega(\tau)$ is constructed, but to merely accept that such a number exists. We then proceed to the extension to this theorem.

Theorem, Extension of Dirichlet's Theorem

Let ξ be a complex number and *N* a positive integer. Then there exists a constant $C = C(\xi, N)$ such that for any positive integer *H*, there exists a non-zero polynomial $P(z) \in \mathbb{Z}[z]$ with degree $\partial P \leq N$ and $h(P) \leq H$ such that $|P(\xi)| < \frac{C}{H^{\sqrt{N-1}}}$.

Corollary

Let $\xi \in \mathbb{C}$ be transcendental. Then there exists a number $\omega(\tau, N)$ such that: ξ is transcendental iff $\omega(\xi) \neq 0$.

Once again, we don't absolutely need to know how $\omega(\tau, N)$ is constructed; what we need to know only that the criterion $\omega(\xi) \neq 0$ is based on an approximation $\Omega(\xi, N, H)$ to ξ that has gotten so close to ξ that the interval $|P(\xi)|$ is incommensurable with zero.

Definition

Let $v(\xi)$ for the least positive integer *N* for which $\omega(\xi, N) = \infty$.

If $\omega(\xi, N)$ is finite for all *N* then $v(\xi) = \infty$ by convention.

If $\omega(\xi)$ is finite, then $v(\xi) = \infty$, but if $\omega(\xi) = \infty$ then $v(\xi)$ may be either finite or infinite.

The Mahler classification

A number	$\omega(\xi) = 0$	$v(\xi) = \infty$
S-number	$0 < \omega(\xi) < \infty$	$v(\xi) = \infty$
T-number	$\omega(\xi) = \infty$	$v(\xi) < \infty$
U -number	$\omega(\xi) = \infty$	$v(\xi) = \infty$

We claim that *S* numbers are infinitesimals. The following theorem is needed.

Theorem for *S* numbers

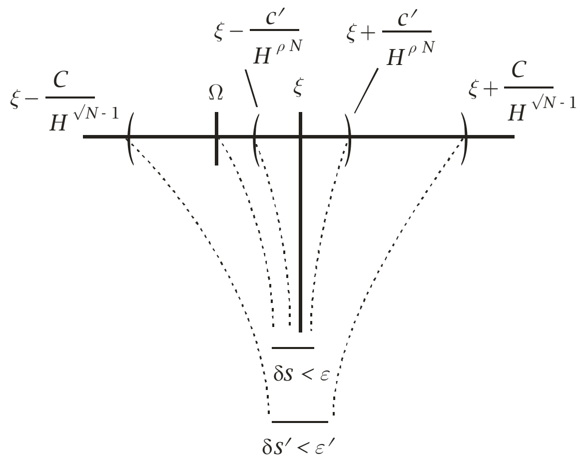
Let $\xi \in \mathbb{C}$ be a transcendental number. Then $0 < \omega(\xi) < \infty$ iff there exists a real number $\rho > 0$ such that for each integer $N \geq 1$, there exists a constant $c' = c'(\xi, N) > 0$ such that for all integers $H \geq 1$ and all polynomials $P(z) \in \underline{P}(H, N)$, the inequality, $\frac{c'}{H^{\rho N}} < |P(\xi)|$ holds.

The constant in the theorem above is explicitly constructed as follows: -

1. Since $0 < \omega(\xi) < \infty$ there exists a $\rho > 0, \rho \in \mathbb{R}$ such that $\omega(\xi, N) < \rho$.
2. Then for fixed *N* and all but finitely many *H*'s we have, $\frac{1}{H^{\rho N}} \leq \Omega(\xi, N, H) \leq |P(\xi)|$.

3. Let there be a list of the exceptions for H to the rule $\frac{1}{H^{\rho N}} \le \Omega(\xi, N, H) \le |P(\xi)|$. Write this list as $H_1 < H_2 < \ldots < H_L$ for some finite L. Let $m = \min\{\Omega(\xi, N, H) : l = 1, 2, \ldots, L\}$.
4. Then $c' = c'(\xi, N) = \min\left\{1, \frac{m}{2} H_L^{\rho N}\right\}$.

This theorem asserts that the growth rate in the Extension to Dirichlet's Theorem cannot be improved. A constant multiple of N is the best possible exponent on H iff $0 < \omega(\xi) < \infty$. S-numbers are transcendental numbers ξ for which $|P(\xi)|$ cannot be made substantially smaller than the upper bound in the Dirichlet Extension Theorem, which is $|P(\xi)| < \frac{C}{H^{\sqrt{N-1}}}$. Combining this with the inner bound of the theorem for S numbers, we obtain: $\frac{C}{H^{\sqrt{N-1}}} > |P(\xi)| > \frac{c'}{H^{\rho N}}$.



In the limit, as $N \to \infty$, the transcendental S number, ξ, is identified with the interval, $\left(\xi - \frac{c'}{H^{\rho N}}, \xi + \frac{c'}{H^{\rho N}}\right)$, and is itself squeezed into a measure incommensurable with zero, $\left(\xi - \frac{C}{H^{\sqrt{N-1}}}, \xi + \frac{C}{H^{\sqrt{N-1}}}\right)$. On the principle that intervals are never destroyed any S number is an infinitesimal. In the limiting process the interval $\left(\xi - \frac{c'}{H^{\rho N}}, \xi + \frac{c'}{H^{\rho N}}\right)$ is never destroyed. One way or another this set must also contain a boundary not identifiable with its interior. Since it is itself squeezed between an open set, we cannot have two disjoint open sets adjacent, without there being a boundary between them, so the boundary that is interior to the outer set is inseparably connected to the interior. Hence, it is an infinitesimal.

The Continuum Hypothesis

So how many points are there in the continuum?

We are nearing the end of our journey – at least for the present time, for there is a lot more one could add. But of course we would like to reflect a little on whether the model of the continuum here presented does create an unambiguous answer to the Continuum Hypothesis. It helps first to reflect on what we have learned about the difference between first-order set theory and analysis.

Set theory as a model building tool for the continuum

Set theory is an incomplete theory of sets, and *a fortiori* of the continuum. There is only one object in set theory that could have any pretention to being the continuum, and that is the Cantor set. A moment's reflection will easily show that there is nothing in ZFC that could possibly resolve the problem of what the continuum is and the relation of the cardinals. The power set axiom generates by cardinal exponentiation all in one go the Cantor set $\mathbf{2}^{\omega}$ from ω, but for that very reason provides no clues as to its inner structure. The other option is to go from the bottom up and create bigger sets out of smaller ones; for this purpose we have the set unions and the axiom of replacement. Taken altogether they make up the constructible sets. We can also create definable sets using first order quantifiers, but of course neither constructible nor definable sets determine the structure of the continuum, so they are in this respect a kind of red-herring.

The ultimate resolution of the structure of the continuum must be derived from physics

Another point that is becoming crystal clear is that the continuum is an empirical object. It does incorporate concepts drawn from phenomenological experience, but these are insufficient to create a clear mathematical intuition, and in all events, even if we did have a clear and distinct idea given to us by intuition, we could still argue that it constituted merely a hypothesis for empirical science. It is in the nature of empirical science to choose between different models of given phenomena. The data for the continuum will come ultimately from physics, and the relation of physics to mathematical physics and thence to pure mathematics is bound to pass through many different layers; mathematical concepts about the continuum will be deeply embedded in approaches to theoretical physics that it will not be easy to subject them to experimental confirmation. Nonetheless, that changes to the way we think about space are taking place seems highly likely.

Set theory as a tool for building models of the continuum

Within this context first-order set theory does provide a unique tool for model building. We now have a multiplicity of models of the continuum to choose from. However, those options have been made sufficiently perspicuous for practical applications to follow – there is a job to do there to bring down the researches into the "higher infinite" to a much more accessible level. But certainly at the theoretical level, first-order set theory does its job well.

A philosophical bias

There is philosophical bias as well. A certain amount of conceptual thinking must take place in science, and not all of this can be reduced to first-order rules. Even if it could (and it cannot), one would expect a dialectical principle to be at work, so that different approaches to the philosophical issues surrounding mathematics and its application to reality would run concurrently. We don't really see anything like that in the present cultural environment. On the contrary seriously good mathematicians and set theorists come up with statements like "all mathematics is set theory" that are just palpably false. In the shadows of this misconception is an even more dangerous one – It is terribly dangerous if one thinks "one is right", for that gives you the authority to legislate – so we see seriously intended and impassioned prescriptions to the effect that whole parts of mathematics should be put in the bin.

Out of the comfort zone

When we step outside the confines of first-order theory we find ourselves in a kind of no man's land. In first order theory we are safe. Even if the theory is not complete, we have the safety of axioms, postulates, definitions, rules of inference – so if you prove something in such a theory, then you won't have an argument with your colleagues – at least they won't be arguing about your results – they might argue about their significance. On the other hand, in the no man's land, you find yourself engaging in all the forbidden practices – thinking in geometric terms; appealing to phenomenology, concepts and intuition; demonstrating solutions as opposed to "rigorously proving them" (that is, in a first order system); and you might also just be talking a whole load of nonsense. Well there are risks in everything.

The Axiom of Completeness and the Continuum Hypothesis

Every distinct model of the continuum in first-order set theory is associated with both a concept of forcing and with a chosen set of new axioms. So in other words, as with the forcing, so with the continuum. What kind of forcing is allowed subject to the Completeness Axiom? **The answer to this question is shockingly simple**. The Completeness Axiom simple states at most a forcing principle that is a subset of Cohen forcing. We look at it in the Cantor form:

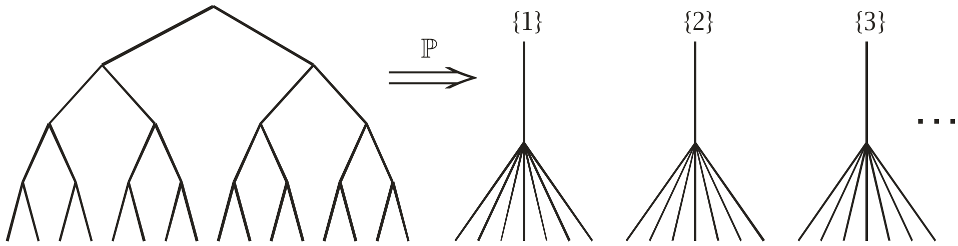
> **The completeness axiom: Cantor's nested interval principle**
>
> Given any nested sequence of closed intervals in $\mathbb{R}$, $[a_1,b_1]\supseteq[a_2,b_2]\supseteq\dots\supseteq[a_n,b_n]\supseteq\dots$, there is at least one real number contained in all these intervals: $\bigcap_{n=1}^{\infty}[a_n,b_n]\neq\varnothing$.

This is self-evidently a form of Cohen forcing: $\mathbb{C}=(\mathrm{Fn}(\omega,2),\subseteq)$. The nested intervals constitute the Poset, and we also have implicit the generic forcing condition that any finite termination of the sequence must lead to a contradiction. For if at some finite stage it would not be possible to find a sub-interval of $[a_n,b_n]$ we would have $a_n=b_n$ and an interval of positive measure would have a zero measure – a contradiction. So the sequence must run on for ever and only be completed in the actually infinite limit by converging on a point that shall be identified with a real number. The Axiom also has the benefit of bringing the real number into existence through the existential quantifier. No other form of constructing real numbers is provided – it is this and this alone. **So no other forcing whatsoever**. **So the Continuum Hypothesis is actually trivially true for the Axiom of Completeness.** Howver, we can also prove the following theorem:

> **Forcing theorem**
>
> All transcendental numbers of the Mahler classification correspond to ultrafilters (functions) produced by Cohen forcing. These ultrafilters are filters isomorphic to **Fin** that are embedded within a subset of the Derived set $2^{\mathbb{N}_\lambda}$ of the actually infinite skeleton of the continuum.[43]

[43] **Proof of the Forcing Theorem.** Let ξ be transcendental. In the Extended Dirichlet Theorem, we construct a series of approximations $P(\xi)_{N,H}$ where $\Omega(\xi,N,H)=\min\{|P(\xi)|:P(z)\in\underline{P}(N,H),P(\xi)\neq 0\}$

Cohen forcing is a device for transforming the Cantor set of boundary points into what is called a root system. The root of each branch of this is a unique rational number, and the branches are the collection of Cohen reals in the null set of the null-meagre decomposition.



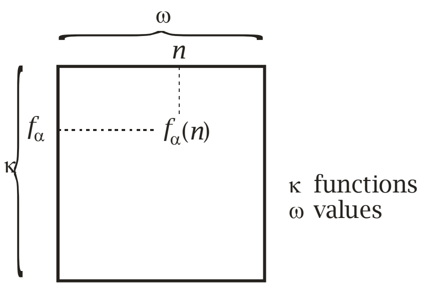
The possibility exists within set theory of different root systems. However we now we show that there is no other kind of forcing that would falsify the Continuum Hypothesis compatible with the Axiom of Completeness. The proof is by contradiction.

Proof

Suppose the Completeness Axiom entails the existence of a forcing: $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$ that generates an ultrafilter. It defines a characteristic function: -

$$f_\alpha(n) = \begin{cases} 1 & \text{if } f_\alpha(n) \text{ is defined and } f_\alpha(n) \in G \\ 0 & \text{if } f_\alpha(n) \text{ is undefined or } f_\alpha(n) \notin G \end{cases}$$

So it is a coding for a real number $x \subset \omega$ that maps finite partial functions $\kappa \times \omega$ to 2.[44]



The mapping defines a sequence $f_\alpha(n) : \kappa \to 2^\omega$. There are ω assignments for each of the κ functions and $\mathrm{Im}(f) \subseteq 2^\omega$. The simplest such function is $\mathbb{P} = \mathrm{Fn}(\omega_2 \times \omega, 2)$ which entails $2^\omega \geq \omega_2$. To make the argument clearer, let us choose a definite example: let $2^\omega = \omega_2$, so we

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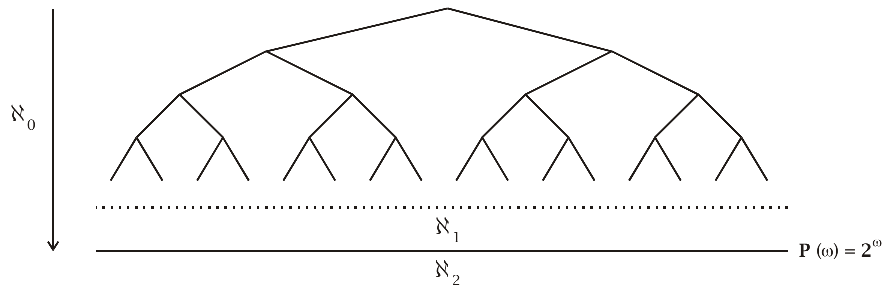
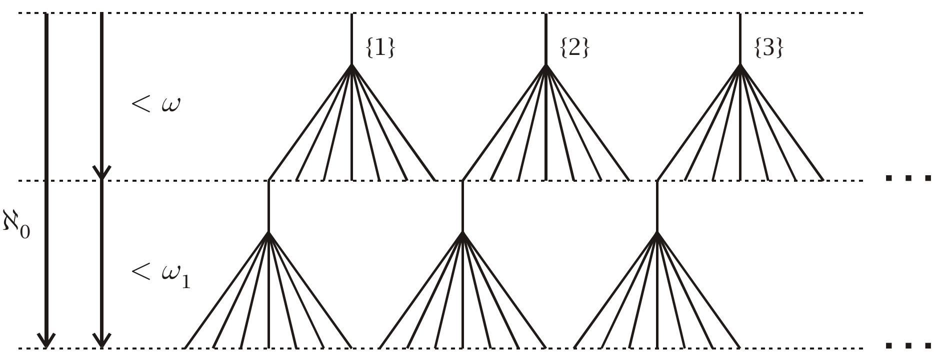
and $\omega(\xi, N, H) = -\dfrac{\log \Omega}{N \log H}$. Since ξ is transcendental, we also have $\omega(\xi) \neq 0$. $P(\xi)_{N,H}$, is a series of algebraic codes that correspond to lattice points within an ideal isomorphic to $\mathbf{Fin} \cong \mathbf{2}^{<\omega}$. The conditions, $\Omega(\xi, N, H) = \min\left\{ |P(\xi)| : P(z) \in \underline{P}(N,H), P(\xi) \neq 0 \right\}$, $\omega(\xi, N) = \lim\left\{ \sup_{H \to \infty} \omega(\xi, N, H) \right\}$ and $\omega(\xi) = \lim\left\{ \sup_{N \to \infty} \omega(\xi, N) \right\}$ together constitute a set of forcing conditions and define a poset, $\mathbb{P}(\xi)$. This poset is open, dense and at each finite level the assumption that ξ is defined by a polynomial immediately leads to the contradiction: $\omega(\xi) = 0$. The conditions, $\omega(\xi) = 0$ and $\omega(\xi) \neq 0$ determine incompatible lattice points at any finite level. Therefore, the poset $\mathbb{P}(\xi)$ defines a generic ultrafilter within an ideal isomorphic to $\mathbf{Fin} \cong \mathbf{2}^{<\omega}$. Every chain has ordinal length $< \omega$, and hence is of order type ω. Therefore, the forcing $\mathbb{P}(\xi)$ is Cohen forcing $\mathbb{P}(\xi) : \omega \to 2$.

[44] One would "prefer" to see $\mathrm{Fn}(\kappa, 2^\omega)$,, $f_\alpha : \kappa \to \mathbf{2}^\omega$ to give it as a mapping from κ into 2^ω but this is not possible because in the ground model the collection of functions is not necessary total, so in the ground model it is possible that $\kappa \geq 2^\omega$.

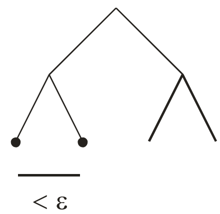
have three cardinalities: $\aleph_0 < \aleph_1 < \aleph_2 = 2^{\aleph_0}$. This is also a model of the continuum which a skeleton of ω atoms[45]. Then the Cantor tree has the distinct families of subsets of all three cardinalities: $\aleph_0 < \aleph_1 < \aleph_2 = 2^{\aleph_0}$. We may show this diagrammatically as follows:



Before the Cantor tree has reached down to the level of actually ω iterations, it has generated a distinct set of width $\aleph_1 < \aleph_2$. At the ωth iteration, the width is $\aleph_2 = 2^{\aleph_0}$. As the tree has skeleton has ω atoms, the tree structure may be transformed into a structure that is visualised as:



Another way of understanding this point is to refer back to the image we made of a point of zero measure:



In this diagram we imagine that the "last branches" of the binary tree that generates the dense subset of the continuum are so close together that the space cannot be measured as difference from zero. In the Mahler classification this space is then equated to a point of zero measure – **it is the space that becomes this point** – not the branches. But the poset $\mathbb{P} = \mathrm{Fn}(\omega_2 \times \omega, 2)$ populates this space with another set of boundary points, thus destroying the Mahler classification. In this model[46] the Mahler classification of transcendental numbers is

[45] It satisfies the countable chain condition.
[46] $\mathrm{MA}(\kappa)$ is the statement: Whenever $(P, \leq)$ is a non-empty c.c.c. partial order, and **D** is a family of $\leq \kappa$ dense subsets of *P*, then there is a filter *G* in *P* such that $(\forall D \in \mathbf{D})(G \cap D \neq 0)$. MA is the statement

not valid. For example, any Liouville number in the Maher classification is the root of a family of real numbers and not a real number itself. It might be called "quasi" algebraic, as in this model we are extending the notion of an algebraic number beyond polynomials of degree ω. To generalise this argument we can replace the definite ordinal ω_2 by an arbitrary $\kappa > \omega$. Hence:

$$\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2) \Rightarrow \neg \text{ Mahler classification} \Rightarrow \neg \text{ Axiom of Completeness}$$

Hence, given the Axiom of Completeness, we have $\neg\, \mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$ for all $\kappa > 1$.

Limitation on the number of data-types

The theory of ordinals is a theory of **data-types**. For example, the ordinal $\omega + 1$ is the data type formed by taking all of the natural numbers as a completed sequence, and then adding another single item of information to that collection. The data-type $\kappa \times \omega$ is an array or matrix of κ rows and ω columns. The axiom of **completeness** gives no information on how to take the limit points of a sequences defined by a partial order of finite subsets of such elements, and explicitly constructs a real number as the limit point of a sequence of data-type ω. **The assumption that there are other data types destroys the skeleton on which it is constructed**.

The relationship between the skeleton and the lattice derived by exponentiation from it

We need to understand the relationship between the skeleton of the line and its exponential derived set. **This is an exact derivation.** In other words, given the skeleton there is only one structure to the continuum, and vice-versa, given a structure on the continuum, there is only one skeleton.

* If in fact we introduce a new data-type into the skeleton, then the Heine-Borel theorem will fail. It will be possible to construct infinite covers of the unit interval using such data-types for which there is no one-point compactification. So we see that the Axiom of Completeness brings the one-point compactification of the unit interval into existence, and in so doing excludes all forcings save Cohen forcing. Hence, **the Continuum Hypothesis is trivially true for the Completeness Axiom**.

The empirical character of the question

Set theory is a tool that opens up the possibility of a vast variety of different structures to the real line. It is prolix and underdetermined. Each structure is characterised by a different set of additional axioms. *The validity of the Completeness Axiom will become an empirical question.* Within first-order set theory all sorts of alternative models are possible.[47] Consider, for example, two illustrations from Kunen [1980] of the use of iterative Cohen forcing to produce alternative models of the continuum.

1. $\mathbb{P} = \mathrm{Fn}(\omega, \kappa)$

$(\forall \kappa < 2^{\omega})(\mathrm{MA}(\kappa))$. $\mathrm{MA}(\kappa)$ calls into existence the cardinal $\kappa < 2^{\aleph_0}$. This tree structure requires Martin's Axiom in the form $\mathrm{MA}(\omega_1)$. As $\aleph_2 = 2^{\aleph_0}$, we have $\neg\, \mathrm{MA}(\omega_2)$.

[47] Kunen [1980] explains this point as follows: "The particular axioms of set theory that $M[G]$ satisfies beyond ZFC will be very sensitive to combinatorial properties satisfied by $\mathbb{P}$ in *M*; most of these properties are *not* absolute. For example, consider the c.c.c. If *M* is a c.t.m. and $\mathbb{P} \in M$, then in **V**, $\mathbb{P}$ is countable and thus trivially has the c.c.c. But $\mathbb{P}$ may well fail to have the c.c.c. in *M*."

Here $\mathrm{Fn}(\omega,\kappa)$ denotes the set of all partial functions from ω into κ. Since κ is uncountable, this poset does not satisfy the countable chain condition.[48] In the ground model, *M*, we have $\kappa > \omega$. In the generic extension $M[G]$ we have $\kappa = \omega$. This forcing is used to show that the notion of cardinality is not absolute for *M*, or absolute in ZFC. A set may have cardinality $\kappa > \omega$ in the ground model, and cardinality $\kappa = \omega$ in the generic extension $M[G]$.

2. $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$

We examined this above; it satisfies the countable chain condition. In the ground model, *M*, we have $\kappa > \omega$ and κ, 2^{ω} are incomparable. In the generic extension $M[G]$ we have $\omega < \kappa = 2^{\omega}$. Since κ can be anything, this falsifies the Continuum Hypothesis.

However, under the Completeness Axiom, the forcings $\mathbb{P} = \mathrm{Fn}(\omega,\kappa)$ and $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$ do not exist. Set theory is a language for describing a multiplicity of different structures of the real line; the Axiom of Completeness picks out just one of these structures, actually the simplest available.

The set of order types of well-orderings of the rational numbers

The following lemma tells us what the ordinal ω_1 represents.

Lemma – Halbeisen [2012]

For every ordinal $\alpha \in \omega_1$ there is a set of rationals $Q_\alpha \subseteq \mathbb{Q} \cap (0,1)$ and a bijection $h_\alpha : \alpha \to Q_\alpha$ such that for all $\beta, \beta' \in \alpha$, $\beta \in \beta' \Leftrightarrow h_\alpha(\beta) < h_\alpha(\beta')$.

Corollary

ω_1 is the set of order types of well-orderings of $\mathbb{Q}$.

This means, $\omega_1 = \mathbf{P}(\mathbb{Q}) = \mathbf{2}^{\eta}$, where η represents the order-type of an element of $\mathbb{Q}$ - a dense, linear order with neither first nor last element. However, the lemma tells us that in order to generate ω_1 we must first give $\mathbb{Q}$ a well-ordering using the Axiom of Choice. The moot question is how the collection of all well-orderings of $\mathbb{Q}$ is to be determined. The prolixity of set theory means that within set theory this is every bit as undetermined a question as the question of the structure of the continuum itself: to every distinct model of the continuum there corresponds a different interpretation of what a well-ordering of $\mathbb{Q}$ might be, and hence the structure of ω_1. The effect of the one-point compactification of the unit interval is to collapse the dense skeleton $\mathbb{Q}$ onto its equinumerous counterpart $\mathbb{N}$. This is because we divide the open interval $[0,1]$ not into a cover of data type ordinal η, but of data type ordinal ω. The partition of $[0,1]$ is into a dense set of ω pieces. So we embed $\mathbf{2}^{\mathbb{Q}}$ within $\mathbf{2}^{\mathbb{N}}$.

Theorem, the Completeness Axiom entails the Continuum Hypothesis

[48] For this reason, it is not a possible model of the continuum subject to the Axiom of Completeness. So we did not consider it in the contradiction argument given above.

The canonical representation of the skeleton is $\mathbb{N}_\lambda = \mathbb{N}_\mathbb{A} \cup \{\lambda\}$. The potentially infinite part of this denoted $\mathbb{N}_\mathbb{A}$ may be placed in one-one correspondence with any potentially infinite dense subset of $\mathbb{R}$, denoted $\mathbb{Q}$. So the skeleton is also of the form $\mathbb{Q}_\lambda = \mathbb{Q}_\mathbb{A} \cup \{\lambda\}$ Under the Axiom of Completeness, the partition is a bijection $\mathbb{Q}_\lambda \to \mathbb{N}_\lambda$. Then $\text{card}\left(2^{\mathbb{Q}_\lambda}\right) = \text{card}\left(2^{\mathbb{N}_\lambda}\right)$. We have $\mathfrak{c} = \text{card}(\mathbb{R}) = \text{card}\left(2^{\mathbb{N}_\lambda}\right) = 2^{\aleph_0}$. Whence $\omega_1 = \text{card}\left(\mathbf{2}^{\mathbb{Q}}\right) \le \text{card}\left(\mathbf{2}^{\mathbb{Q}_\lambda}\right) = \text{card}\left(\mathbf{2}^{\mathbb{N}_\lambda}\right) = 2^{\aleph_0} = \mathfrak{c}$. Since , $\mathfrak{c} = 2^{\aleph_0} \ge \omega_1$, we have $\mathbf{2}^{\aleph_0} = \aleph_1$.

On the differences between C19th and C20th mathematics

The Axiom of Completeness is second order and quantifies over properties of individuals. To quantify over something is to begin to accept that thing as "real" – as part of the world. But properties seem to belong to the intellectual world of subjective interpretation rather than to the "real world" of things. In the late C19th idealism was the dominant philosophy, and the interweaving of *ideas* with *things* would not have appeared strange or problematic, had the matter been considered. But in the early C20th positivism overwhelmed idealism, and the idea of intellectual concepts being integral of science came to be seen as problematic. Second order properties were dropped and the spirit of the age demanded that science be conducted in first-order logic. The notion of the actual infinite expressed in the Axiom of Completeness is problematic. As an intellectual structure, or system of explanation in its own right, it poses one set of problems, but once it is projected back onto reality, or nature, another. If a point is merely an intellectual *idea* that does not correspond to an existent entity in nature, it requires no further leap of faith when *imagining* an actually infinite collection of them. But that a particle in motion should pass through an actually infinite collection of points in nature conceived as a *reality independent of human thought* is repugnant to understanding, or, at least, may seem so. So for various reasons, by no means all explicit, the spirit of the age has legislated against second order concepts. Mathematics still proceeds according to both traditions. Currently, there is a movement against real numbers. The actual infinite, in the form of the ordinal ω, is less problematic than second-order quantification, for ω is a *finite symbol* integrated into a first-order system, so can be treated as part of a formal system. It's difficult to read the ontological commitment of the actual infinity that is implied by the Axiom of Completeness in this way. Possibly, we are collectively experiencing a cultural revolution akin to that initiated by Aristotle when he legislated against the actual infinite. The alternative, that is, the counter-revolution, would seem to be to drop the preoccupation with first-order conceptions and revert to an interpretation of science and reality that does not exclude human understanding as a primitive.

Melampus

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